University of Miskolc

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Name:...............................

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NUMERICAL AND OPTIMIZATION METHODS

Second test

1 (20 pts) Find the minimum point of the function

$$f(x, y) = 4x^2 - 8xy + 2y^2 + 10x - 6y + 12$$

by Newton method, with guess vector

$$\mathbf{x_1} = [0, 0].$$

$$\nabla f = \begin{bmatrix} 8x - 8y + 10 \\ -8x + 4y - 6 \end{bmatrix} \qquad \nabla f(x_1) = \begin{bmatrix} 0 - 0 + 10 \\ -0 + 0 - 6 \end{bmatrix} = \begin{bmatrix} 10 \\ -6 \end{bmatrix}$$

$$Hf = \begin{bmatrix} 8 & -8 \\ -8 & 4 \end{bmatrix} \qquad Hf(x_1) = \begin{bmatrix} 8 & -8 \\ -8 & 4 \end{bmatrix}$$

$$Hf(x_1) \cdot s_1 = -\nabla f(x_1) \qquad s_1 = \begin{bmatrix} \cancel{9} \\ \cancel{6} \end{bmatrix}$$

$$\Downarrow$$

I. $8a - 8b = -10$

II. $-8a + 4b = 6$

I. + II. $\quad -4b = -4$

$$\underline{\underline{b = 1}}$$

I. $8a - 8 = -10 \quad / + 8$

$8a = -2 \quad / : 8$

$$\underline{a = -1/4}$$

$$s_1 = \begin{bmatrix} -1/4 \\ 1 \end{bmatrix}$$

$$x_2 = x_1 + s_1 = \begin{bmatrix} -1/4 \\ 1 \end{bmatrix}$$

$$\nabla f(x_2) = \begin{bmatrix} -2 - 8 + 10 \\ 2 + 4 - 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \underline{\underline{\begin{bmatrix} -1/4 \\ 1 \end{bmatrix}}}$$

minimum point.

2 (20 pts) Find the minimum point of the function

$$f(x, y) = 4x^2 - 8xy + 2y^2 + 10x - 6y + 12$$

by gradient method, with guess vector

$$\mathbf{x_1} = [0, 0].$$

Determine the next point $\mathbf{x_2}$.

$$\nabla f = \begin{bmatrix} 8x - 8y + 10 \\ -8x + 4y - 6 \end{bmatrix}$$

$$\nabla f(x_1) = \begin{bmatrix} 10 \\ -6 \end{bmatrix} \longrightarrow d_1 = -\nabla f(x_1) = \begin{bmatrix} -10 \\ 6 \end{bmatrix}$$

$$\varphi(\lambda) = f(x_1 + \lambda d_1) = f((-10\lambda, 6\lambda)) =$$

$$= 4(-10\lambda)^2 - 8(-10\lambda) \cdot 6\lambda + 2(6\lambda)^2 + 10(-10\lambda)$$

$$-6 \cdot 6\lambda + 12 = 400\lambda^2 + 480\lambda^2 + 72\lambda^2 - 100\lambda$$

$$-36\lambda + 12 = 952\lambda^2 - 136\lambda + 12 \longrightarrow \text{min}$$

$$\varphi'(\lambda) = 1904\lambda - 136 = 0$$

$$\lambda_1 = \frac{136}{1904} = \underline{\frac{1}{14}}$$

$$x_2 = x_1 + \lambda_1 d_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \frac{1}{14} \begin{bmatrix} -10 \\ 6 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} -5/7 \\ 3/7 \end{bmatrix}$$

3 (5 pts) Minimize the function

$$f(x, y) = x^2 + y^2$$

subject to:

$$x + 2y = 4 \longrightarrow h(x,y) = x + 2y - 4 = 0$$

and

$$2x + 3y \leq 7 \longrightarrow g(x,y) = 2x + 3y - 7 \leq 0$$

by Karush-Kuhn-Tucker method. Give the Karush-Kuhn-Tucker conditions only!

$$\nabla f = \begin{bmatrix} 2x \\ 2y \end{bmatrix} \qquad \nabla h = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \nabla g = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 2x \\ 2y \end{bmatrix} + \mu_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \lambda_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\boxed{\nabla f(x^*) + \sum_{j=1}^{p} \mu_j \nabla g_j(x^*) + \sum_{i=1}^{m} \lambda \nabla h(x^*) = 0}$$

$$\Downarrow$$

I. $2x + 2\mu_1 + \lambda_1 = 0$

II. $2y + 3\mu_1 + 2\lambda_1 = 0$

$\boxed{\mu_j \cdot g_j(x^*) = 0} \longrightarrow$ III. $\mu_1 (2x + 3y - 7) = 0$

IV. $x + 2y - 4 = 0$

$\boxed{h_i(x^*) = 0}$

$\lambda_1, \mu_1 \in \mathbb{R}$

$\mu_1 \geq 0.$

KKT conditions

4 (5 pts) Solve the initial value problem:

$$f(x,y)$$

$$y' = x + 2y$$

$$y(0) = 0$$

numerically by Euler-method. Finding a value for the solution at $x = 1$ and using steps of size $h = 0.25$.

$$x_0 = 0$$
$$x_1 = 0 + 0.25 = 0.25$$
$$x_2 = 0.25 + 0.25 = 0.5$$
$$x_3 = 0.5 + 0.25 = 0.75$$
$$x_4 = 0.75 + 0.25 = \underline{1}$$

$$y_{n+1} = y_n + h \cdot f(x_n, y_n)$$

$$\underline{\underline{y_0 = 0}}$$

$$y_1 = y_0 + h\,(f(x_0, y_0)) = 0 + 0.25 \cdot (0 + 2 \cdot 0) = \underline{\underline{0}}$$

$$y_2 = y_1 + h\, f(x_1, y_1) = 0 + 0.25\,(0.25 + 2 \cdot 0) = \underline{\underline{0.0625}}$$

$$y_3 = y_2 + h\, f(x_2, y_2) = 0.0625 + 0.25\,(0.5 + 2 \cdot 0.0625) = \underline{\underline{0.21875}}$$

$$y_4 = y_3 + h\, f(x_3, y_3) = 0.21875 + 0.25\,(0.75 + 2 \cdot 0.21875) = \underline{\underline{0.515625}}$$

$$\Downarrow$$

$$\underline{y(1) \approx 0.515625}$$