Classical error analysis

Introduction

During the practical, technical calculations we rarely have accurate data at hand; we mostly must calculate with approximate values. Inaccuracy of the samples results inaccurate result values. The classical error analysis deals primarily with the spreading of the errors hidden in the input data. They affect the calculation results, their impact exerted to the calculation results, since the approximations are viable only if we know some upper bound of the error. Fortunately, such bounds usually can be determined.

Let x be an accurate value, and a the approximation of x. With the usual denotation: $x \approx a$.
The $x - a$ difference is called the **error** of the a approximation.

The $|x - a|$ number is called the **absolute error** of the a approximation.

The Δa value (which stands for $|x - a| \leq \Delta a$) is called the **bound of the absolute error** of the approximation a.

The relative error of some a approximate value of number x is the $\frac{|x-a|}{|x|}$ amount. We call the δa value (which stands for $\frac{|x-a|}{|x|} \leq \delta a$), the relative error bound of the a approximation. Since the accurate x value is usually not known, therefore we often use the approximation $\delta a \approx \frac{\Delta a}{|a|}$ for the relative error bound.

Remarks:

1) $|x - a| \leq \Delta a \Leftrightarrow x \in [a - \Delta a, a + \Delta a]$, therefore we often use the

$x = a \pm \Delta a$ reference as well.

2) The absolute error bound does not give any information about the real accuracy of the approximation, therefore we introduce the concept of relative error, which compares the error to the accurate value. For example, the same approximation with a 0.05 absolute error bound means an entirely different accuracy if we are talking about the approximation of a hypothetically 1000 magnitude value then if the magnitude of the approximated value is 0.001.

3) In the practice (inaccurately according to the definitions above) people refer to the absolute error bound as absolute error and to the relative error limit as relative error.

4) The relative error is a quantity without a unit of measurement; its value can also be given in percentage.

Example:

Let the accurate value be $x = \pi$. Its most frequently used approximation: $a = 3.14$. We cannot define the extent of the $x - a$ error and the $|x - a|$ absolute error, since π is an irrational number. We do not know its accurate value. We can estimate the upper bound of the absolute error. It will be 0.005 in case of the values rounded to two decimals. Thus the absolute error bound of the approximation: $\Delta a = 0.005$, which means that the accurate value of π deviates at most by five thousands from the 3.14 approximate value: $|\pi - 3.14| \leq 0.005$, so $\pi \in [3.135; 3.145]$, with the regular notation: $\pi = 3.14 \pm 0.005$.

We give the relative error bound (relative error) of the approximation with the quotient of the absolute error bound and the approximation:

$$\delta a \approx \frac{\Delta a}{|a|} = 0.0053.14 \approx 0.00159 \approx 0.16\%$$

The absolute error bounds of the basic operations

Let x and y be accurate values, their approximations: $x \approx a$ and $y \approx b$, the absolute error bounds of the approximation in turn Δa and Δb respectively, namely $|x - a| \leq \Delta a$ and $|y - b| \leq \Delta b$. The equations below give the absolute error bounds of the result of base operators.

Theorem:

$$\Delta(a + b) = \Delta a + \Delta b$$

$$\Delta(a - b) = \Delta a + \Delta b$$

$$\Delta(ab) \approx |a| \cdot \Delta b + \Delta a \cdot |b|$$

$$\Delta\left(\frac{a}{b}\right) \approx \frac{|a| \cdot \Delta b + \Delta a \cdot |b|}{|b|^2}$$

Proof:

Addition: (based on the triangle inequality)

$$|(x+y)-(a+b)| = |(x-a)+(y-b)|$$

$$\leq\ |x-a|+|y-b| \leq \Delta a + \Delta b$$

Subtraction:

$$|(x-y)-(a-b)| = |(x-a)-(y-b)|$$

$$\leq |x-a|+|y-b| \leq \Delta a + \Delta b$$

Multiplication:

$$|xy-ab| = \left|\big((x-a)+a\big)\big((y-b)+b\big)-ab\ \right|$$

$$=\ |(x-a)(y-b)+(x-a)b+a(y-b)+ab-ab|$$

$$\leq \Delta a \cdot \Delta b + |b| \cdot \Delta a + |a| \cdot \Delta b \approx |a| \cdot \Delta b + |b| \cdot \Delta a$$

(we ignore the $\Delta a \Delta b$ second order error member)

Division: In the case of division, we obviously suppose that the denominator is not zero and we obtain

$$\left|\frac{x}{y}-\frac{a}{b}\right| = \left|\frac{a+\Delta a}{b+\Delta b}-\frac{a}{b}\right| = \left|\frac{-a\cdot\Delta b + b\cdot\Delta a}{b(b+\Delta b)}\right|$$

$$\leq \frac{|a|\cdot|\Delta b|+|b|\cdot|\Delta a|}{b^2\cdot\left|1+\frac{\Delta b}{b}\right|} \leq \frac{|a|\cdot|\Delta b|+|b|\cdot|\Delta a|}{b^2}$$

Here we can ignore the $\frac{\Delta b}{b}$ member next to the 1. This is how we get the statement.

Example:

Let $x = 20 \pm 0.5$, and $y = 5 \pm 0.1$.

Then

$$a = 20, \qquad \Delta a = 0.5$$

and

$$b = 5, \qquad \Delta b = 0.1.$$

In the case of addition and subtraction:

$$\Delta(a+b) = \Delta a + \Delta b = 0.5 + 0.1 = 0.6$$

$$\Delta(a-b)=\Delta a+\Delta b=0.5+0.1=0.6$$

Important: In the case of subtraction, the absolute errors are cumulated as well! Let us think it through. If the value of x is the greatest possible (20.5) and y the value of (4.9), then the difference of the accurate values is 15.6, namely it differs by 0.6 from the difference of the approximation values, from 15. (The situation is the same if x is the minimum, and y is the maximum value.)

In the case of multiplication and division:

$$\Delta(ab)\approx|a|\cdot\Delta b+\Delta a\cdot|b|=20\cdot0.1+5\cdot0.5=4.5$$

$$\Delta\left(\frac{a}{b}\right)\approx\frac{|a|\cdot\Delta b+\Delta a\cdot|b|}{|b|^2}=\frac{20\cdot0.1+5\cdot0.5}{5^2}=0.18$$

Let's examine how much the maximum value can be in case of multiplication! For this both factors have to be maximal, namely $20.5\cdot5.1=104.55$ is the maximum of the multiplication, namely the difference from the multiplication of the approximation values is 4.55, namely the accurate value of the absolute error limit of the multiplication. Our formula gives only the approximation of this value. It can be seen from the proof that we have ignored the multiplication of the error members, in our case the $0.5\cdot0.1=0.05$ amounts, which is just the difference of the result calculated in the two ways. In practice generally there is no other way to accurately calculate the errors (and usually it is enough to give only their magnitude), therefore we settle for the approximate results given in the formula.

Remark:

The absolute error bound of the division in case of b close to 0 can be extremely large, therefore our algorithms must be adjusted in such a way that we divide with the number with the greater absolute value!

The relative error bounds of the basic operations

The relative error bounds of basic operations can be obtained, if we divide the absolute error bound for the operation by the absolute value of the approximation value to the operation. In case of addition, however, instead of the

$$\frac{\delta(a+b)}{|a+b|}=\frac{(\Delta a+\Delta b)}{|a+b|}$$

correlation we use a rougher estimation, using the fact that the relative error of the value cannot be greater than the greatest of the relative errors of the members. In the other operations, we apply the definition of the relative error, and simplify where possible.

Theorem:

$$\frac{\Delta(a+b)}{|a+b|}=max\left\{\frac{\Delta a}{|a|},\frac{\Delta b}{|b|}\right\}$$

$$\frac{\Delta(a-b)}{|a-b|}=\frac{\Delta a+\Delta b}{|a-b|}$$

$$\frac{\Delta(ab)}{|ab|}\approx\frac{\Delta a}{|a|}+\frac{\Delta b}{|b|}$$

$$\frac{\Delta\left(\frac{a}{b}\right)}{\left|\frac{a}{b}\right|}\approx\frac{\Delta a}{|a|}+\frac{\Delta b}{|b|}$$

Example:
Let $x = 20 \pm 0.5$, $y = -5 \pm 0.1$, $z = 10 \pm 0.2$. Let's calculate the relative error of the $t = \frac{xy}{z}$ quotient!

First solution:
The approximate value of t: $\frac{20\cdot(-5)}{10} = -10$.
The absolute error bound of the quotient: $\frac{|20\cdot(-5)|\cdot 0.2+|10|\cdot(|20|\cdot 0.1+|-5|\cdot 0.5)}{10^2} = \frac{65}{100} = 0.65$. (Because the numerator is a multiplication, therefore when calculating the absolute error of the numerator we used the rule which applies to the multiplication.)
The obtained relative error: $\frac{0.65}{|-10|} = 0.065$.

Second solution:
We can get the result more easily if we take into account that in multiplication the relative errors are summed, so it is enough to sum the relative error of the three variables present in the formula:

$$\frac{0.5}{20} + \frac{0.1}{|-5|} + \frac{0.2}{10} = 0.065.$$

Example:
We measured two resistors with an instrument and we got the following results:
$R_1 = 110.2 \pm 0.3\Omega$, $R_2 = 65.6 \pm 0.2\Omega$. We calculate the original resistance obtained by parallel connection with the known $R_e = \frac{R_1R_2}{R_1+R_2}$ formula. Let's define the relative error bounds of the input data and the R_e original approximation value. Let's calculate the ΔR_e absolute and the $\frac{\Delta R_e}{|R_e|}$ relative error bound of the approximation value in two ways as well:
(i) using only the absolute bounds of the input data the ΔR_e and then the $\frac{\Delta R_e}{|R_e|}$ relative bound,
(ii) using only the relative bounds of the input data the $\frac{\Delta R_e}{|R_e|}$ relative limit and then the ΔR_e absolute bound.

Let's explain why the results are different.

Solution: $\frac{\Delta R_1}{|R_1|} = \frac{0.3}{110.2} = 0.00272$, $\frac{\Delta R_2}{|R_2|} = \frac{0.2}{65.6} = 0.00305$, the original resistance is approximately $R_e = 41.12127$.

(i) $\Delta(R_1R_2) = 0.2\cdot 110.2 + 0.3\cdot 65.6 = 41.72$, $\Delta(R_1+R_2) = 0.3+0.2 = 0.5$, $\Delta R_e = 0.3543$, the relative error is: 0.0086.
(ii) $\frac{\Delta R_e}{|R_e|} = \frac{\Delta R_1}{|R_1|} + \frac{\Delta R_2}{|R_2|} + \max\{\frac{\Delta R_1}{|R_1|}, \frac{\Delta R_2}{|R_2|}\} = 0.00882$, $\Delta R_e = 0.3627$.

The differences arise because of the various omissions and different estimations used during the two calculations.

Exercises:

1. Let $x = 5.24 \pm 0.03, y = 3.74 \pm 0.01$ and q to denote the approximation value of quotient $\frac{x+y}{x-y}$. Determine $q, \Delta q$ and δq amounts.

2. Let $x = 4 \pm 0.05, y = 2.5 \pm 0.02.$ How much is the relative error of the operation $x^2 \cdot y^3$?

3. The parallel sides of one trapeze: $a = 12 \pm 0.6, c = 8 \pm 0.2.$ Its height: $m = 5 \pm 0.1.$ With how great a relative error can we calculate the area?

4. One side of a triangle is $c \approx 10$ cm, the angles lying on it: $\alpha = 30^\circ, \beta = 45^\circ.$ We know that the absolute error bound of the side is 0.05 cm, and that of the angles is $0.1^\circ.$ Define the approximation of the missing angle and the shortest side and their absolute error bound.

5. Let $a = 5 \pm 0.2,$ $b = 3 \pm 0.1,$ $c = 1 \pm 0.4.$ Determine $q, \Delta q$ and δq amounts, where $q = a(b-c)$.

1. $q = \frac{5.24+3.74}{5.24-3.74} \approx 5.9867$

$\Delta q = \frac{|5.24+3.74|\cdot(0.03+0.01)+|5.24-3.74|\cdot(0.03+0.01)}{(5.24-3.74)^2} \approx 0.1863$

$\delta q = \frac{\Delta q}{|q|} = \frac{0.1863}{5.9867} \approx 0.0311$

2. Because $x^2 \cdot y^3 = x \cdot x \cdot y \cdot y \cdot y,$ we have to add the double of the error belonging to the x to the relative error of the multiplication and the triple of the error belonging to the y: $2 \cdot \frac{0.05}{4} + 3 \cdot \frac{0.02}{2.5} = 0.049.$ Using the absolute error limit we calculate the result. Based on the error formula of replacing into the function: $\Delta x^2 \approx 2 \cdot 4 \cdot 0.05 = 0.4$ and $\Delta y^3 \approx 3 \cdot 2.5^2 \cdot 0.02 = 0.375,$ so $\Delta x^2 \cdot y^3 \approx 4^2 \cdot 0.375 + 2.5^3 \cdot 0.4 = 12.25.$ The approximation value of the multiplication: $4^2 \cdot 2.5^3 = 250,$ therefore the relative error is: $\frac{12.25}{250} = 0.049.$

3. The area formula of a trapeze is $t = \frac{(a+c)}{2} \cdot m.$ For the relative error we have to add the relative error of the $a+c$ sum and the m height. (The 2 point value add is present in the nominator, so its relative error is 0.) The result is: $\delta t \approx \max\{\frac{0.6}{12}, \frac{0.2}{8}\} + \frac{0.1}{5} = 0.05 + 0.02 = 0.07$.

4. $\gamma = 180^\circ - (\alpha + \beta) \approx 105^\circ.$ $\Delta\gamma = 0^\circ + (0.1^\circ + 0.1^\circ) = 0.2^\circ.$
Denote with a the angle opposite to α (the shortest side is located opposite to the smallest angle). Based on the sine theorem: $a = \frac{c \cdot sin\alpha}{sin\gamma} \approx \frac{10 \cdot sin30^\circ}{sin105^\circ} \approx 5.1764$

$$\Delta(sin\alpha) \approx |cos\alpha| \cdot \Delta\alpha \approx 0.866 \cdot 0.1 \cdot \frac{\pi}{180} \approx 0.0015 \quad \text{(we calculate in radians!)}$$

$$\Delta(sin\gamma) \approx |cos\gamma| \cdot \Delta\gamma \approx 0.2588 \cdot 0.2 \cdot \frac{\pi}{180} \approx 0.0009$$

$$\Delta(c \cdot sin\alpha) \approx |c| \cdot \Delta(sin\alpha) + |sin\alpha| \cdot \Delta c = 10 \cdot 0.0015 + 0.5 \cdot 0.05 = 0.04$$

$$\Delta a \approx \frac{(|c \cdot sin\alpha| \cdot \Delta(sin\gamma) + |sin\gamma| \cdot \Delta(c \cdot sin\gamma)}{sin^2\gamma} = \frac{5 \cdot 0.0009 + 0.9659 \cdot 0.04}{0.933} \approx 0.0462$$

5. $q = 10, \Delta(q) = 2,9, \delta(q) = 0,29$