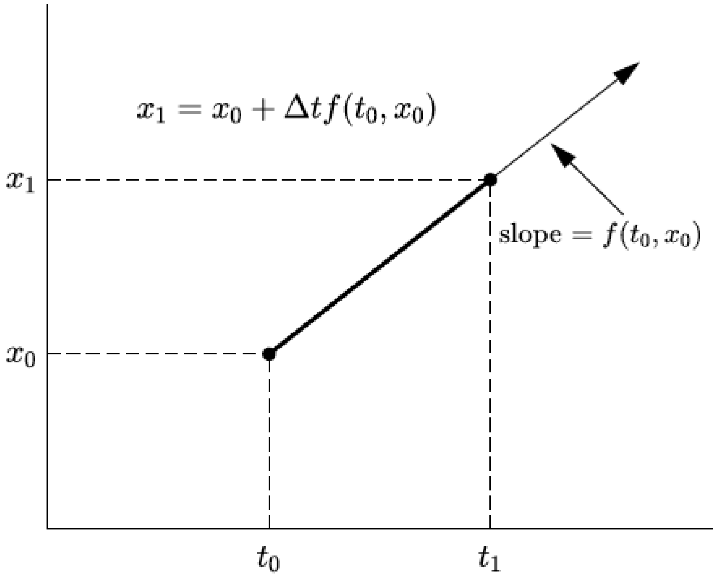
Euler method



The Euler method is the most straightforward method to integrate a differential equation. The first order differential equation $\dot{x} = f\,(t, x)$, with initial condition $x(t_0) = x_0$, provides the slope $f(t_0, x_0)$of the tangent line to the solution curve x = x(t) at the point (t_0, x_0). With a small step size $\Delta t = t_1 - t_0$, the initial condition (t_0, x_0) can be marched forward to (t_1, x_1) along the tangent line (see figure) using $x_1 = x_0 + \Delta t\, f(t_0, x_0)$. The point (t_1, x_1) then becomes the new initial condition and is marched forward to (t_2, x_2) along a newly determined tangent line with slope given by f(t_1, x_1)). This iterative method is usually written as

$$x_{n+1} = x_n + \Delta t\, f\,(t_n, x_n).$$

For small enough Δt, the numerical solution should converge to the exact solution of the ode, when such a solution exists.

The Euler Method has a local error, that is, the error incurred over a single time step, of $O(\Delta t^2)$. The global error, however, comes from integrating out to a time T. If this integration takes N time steps, then the global error is the sum of N local errors. Since $N = T/\Delta t$, the global error is given by $O(\Delta t)$, and it is customary to call the Euler Method a first-order method.

Modified Euler method

The modified Euler method, also called Heun’s method or the predictor-corrector method, is a second order method. The idea is to average the value of $\dot{x}$ at the beginning and end of each time step.

Hopefully, we could write

$$x_{n+1} = x_n + \frac{1}{2}\Delta t\left(f(t_n, x_n) + f(t_n + \Delta t, x_{n+1})\right).$$

The obvious problem with this formula is that the unknown value x_{n+1} appears on the right-handside. We can, however, estimate this value, in what is called the predictor step. For the predictor step, we use the Euler method to find

$$x_{n+1}^{p} = x_n + \Delta t f(t_n, x_n)$$

Then the corrector step becomes

$$x_{n+1} = x_n + \frac{1}{2}\Delta t\left(f(t_n, x_n) + f(t_n + \Delta t, x_{n+1}^{p})\right).$$

The Modified Euler method is usually coded as

$$k_1 = \Delta t f(t_n, x_n),\ \ k_2 = \Delta t f(t_n + \Delta t, x_n + k_1)$$

$$x_{n+1} = x_n + \frac{1}{2}(k_1 + k_2).$$

The modified Euler method is one of a family of methods called second-order Runge-Kutta methods.

Runge-Kutta methods

- To illustrate the derivation of Runge-Kutta methods, we derive here the complete family of second order methods. We march the solution of $\dot{x} = f(t,x)$ forward by writing

$$k_1 = \Delta t f(t_n, x_n),\; k_2 = \Delta t f(t_n + \alpha\Delta t, x_n + \beta k_1)$$

$$x_{n+1} = x_n + ak_1 + bk_2.$$

- where α, β, a and b are constants that define particular second-order methods. We will constrain the values of these constants by computing the Taylor series of x_{n+1} in two ways.
- First, we compute the Taylor series for x_{n+1} directly:
  - $x_{n+1} = x(t_n + \Delta t) = x(t_n) + \Delta t\dot{x}(t_n) + \frac{1}{2}(\Delta t)^2\ddot{x}(t_n) + O(\Delta t^3).$
- Now, $\dot{x}(t_n) = f(t_n, x_n)$. The second derivative is more tricky and requires partial derivatives. We have

$$\ddot{x}(t_n) = \frac{d}{dt} f(t, x(t))\Big]_{t=t_n} = f_t(t_n, x_n) + \dot{x}(t_n) f_x(t_n, x_n) \,|= f_t(t_n, x_n) + f(t_n, x_n) f_x(t_n, x_n).$$

Putting all the terms together, we obtain

$$x_{n+1} = x_n + \Delta t f(t_n, x_n) + \frac{1}{2}(\Delta t)^2 \left(f_t(t_n, x_n) + f(t_n, x_n) f_x(t_n, x_n)\right) + O(\Delta t^3).$$

Second, we compute the Taylor series for x_{n+1} from the Runge-Kutta formula. We start with

$$x_{n+1} = x_n + a\Delta t f(t_n, x_n) + b\Delta t f\left(t_n + \alpha\Delta t, x_n + \beta\Delta t f(t_n, x_n)\right) + O(\Delta t^3);$$

and the Taylor series that we need is

$$f\left(t_n + \alpha\Delta t, x_n + \beta\Delta t f(t_n, x_n)\right) = f(t_n, x_n) + \alpha\Delta t f_t(t_n, x_n) + \beta\Delta t f(t_n, x_n) f_x(t_n, x_n) + O(\Delta t^2).$$

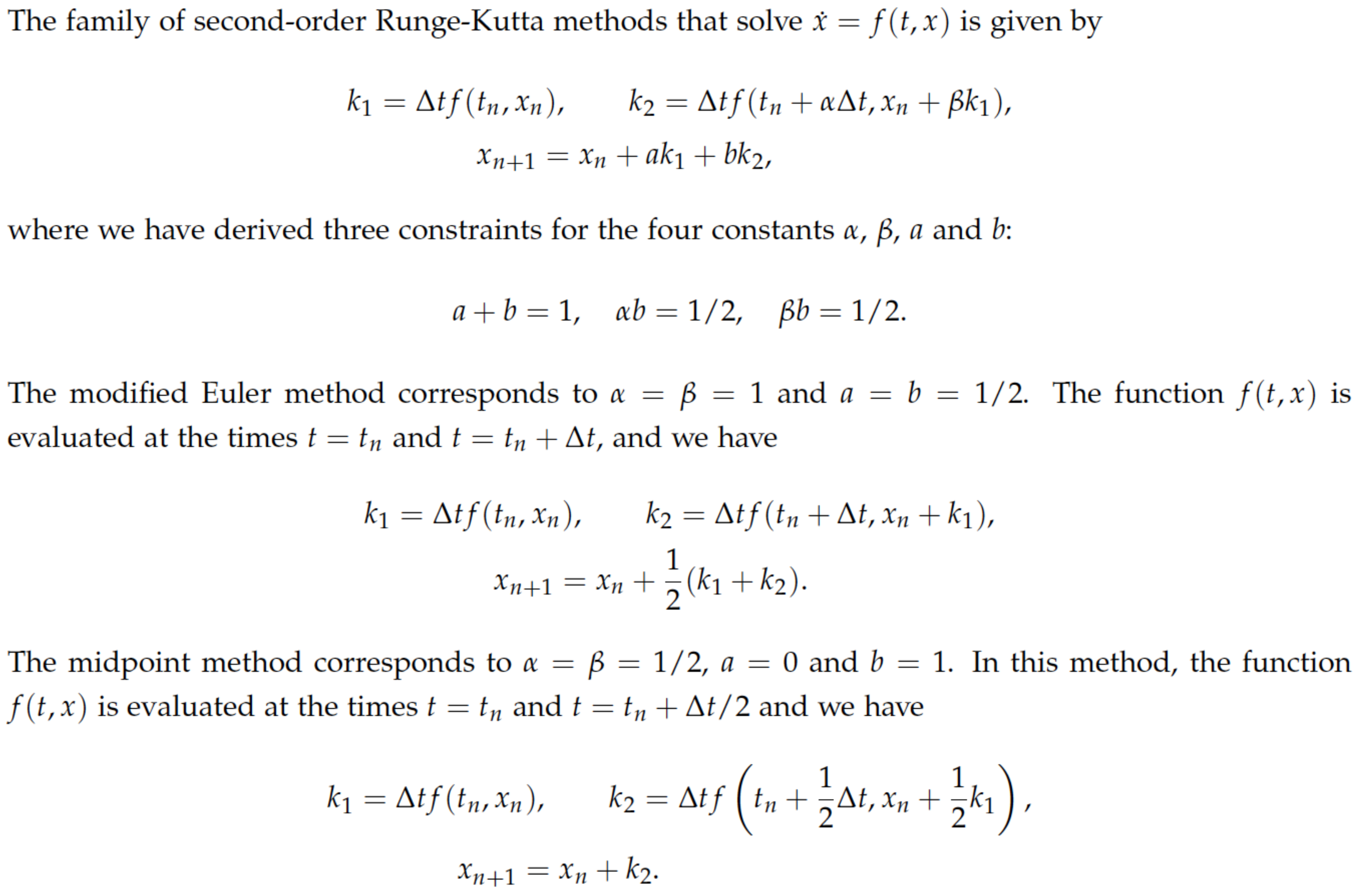
The Taylor-series for x_{n+1} from the Runge-Kutta method is therefore given by

$$x_{n+1} = x_n + (a + b)\Delta t f(t_n, x_n) + (\Delta t)^2 \left(\alpha b f_t(t_n, x_n) + \beta b f(t_n, x_n) f_x(t_n, x_n)\right) + O(\Delta t^3).$$

Comparing (50.1) and (50.2), we find three constraints for the four constants:

$$a + b = 1, \quad \alpha b = 1/2, \quad \beta b = 1/2.$$

Second-order Runge-Kutta methods



The family of second-order Runge-Kutta methods that solve $\dot{x} = f(t, x)$ is given by

$$k_1 = \Delta t f(t_n, x_n), \qquad k_2 = \Delta t f(t_n + \alpha\Delta t, x_n + \beta k_1),$$
$$x_{n+1} = x_n + ak_1 + bk_2,$$

where we have derived three constraints for the four constants α, β, a and b:

$$a + b = 1, \quad \alpha b = 1/2, \quad \beta b = 1/2.$$

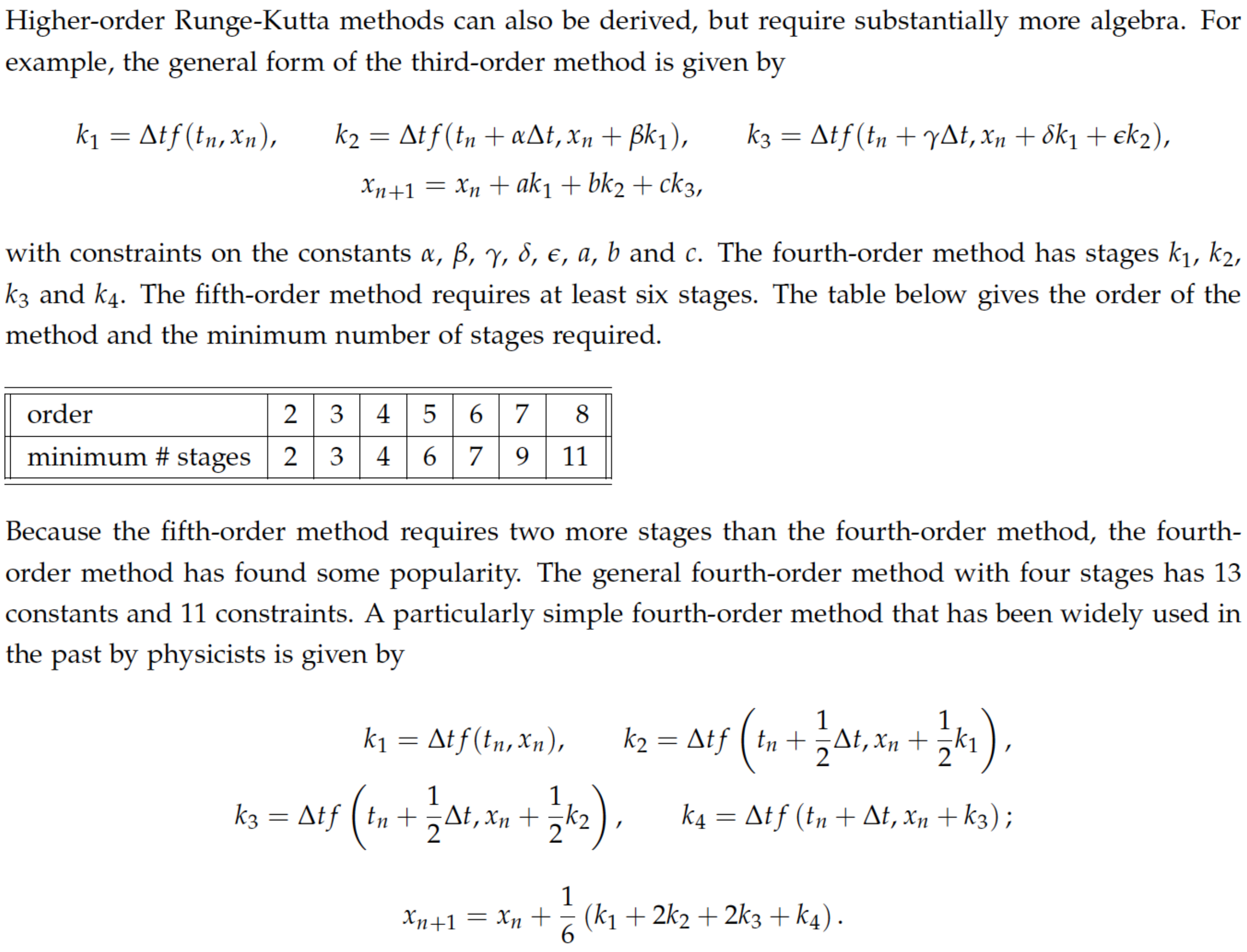
The modified Euler method corresponds to $\alpha = \beta = 1$ and $a = b = 1/2$. The function $f(t, x)$ is evaluated at the times $t = t_n$ and $t = t_n + \Delta t$, and we have

$$k_1 = \Delta t f(t_n, x_n), \qquad k_2 = \Delta t f(t_n + \Delta t, x_n + k_1),$$
$$x_{n+1} = x_n + \frac{1}{2}(k_1 + k_2).$$

The midpoint method corresponds to $\alpha = \beta = 1/2$, $a = 0$ and $b = 1$. In this method, the function $f(t, x)$ is evaluated at the times $t = t_n$ and $t = t_n + \Delta t/2$ and we have

$$k_1 = \Delta t f(t_n, x_n), \qquad k_2 = \Delta t f\left(t_n + \frac{1}{2}\Delta t, x_n + \frac{1}{2}k_1\right),$$
$$x_{n+1} = x_n + k_2.$$

Higher-order Runge-Kutta methods



Higher-order Runge-Kutta methods can also be derived, but require substantially more algebra. For example, the general form of the third-order method is given by

$$k_1 = \Delta t f(t_n, x_n), \qquad k_2 = \Delta t f(t_n + \alpha \Delta t, x_n + \beta k_1), \qquad k_3 = \Delta t f(t_n + \gamma \Delta t, x_n + \delta k_1 + \epsilon k_2),$$
$$x_{n+1} = x_n + a k_1 + b k_2 + c k_3,$$

with constraints on the constants α, β, γ, δ, ϵ, a, b and c. The fourth-order method has stages k_1, k_2, k_3 and k_4. The fifth-order method requires at least six stages. The table below gives the order of the method and the minimum number of stages required.

order	2	3	4	5	6	7	8
minimum # stages	2	3	4	6	7	9	11

Because the fifth-order method requires two more stages than the fourth-order method, the fourth-order method has found some popularity. The general fourth-order method with four stages has 13 constants and 11 constraints. A particularly simple fourth-order method that has been widely used in the past by physicists is given by

$$k_1 = \Delta t f(t_n, x_n), \qquad k_2 = \Delta t f\left(t_n + \frac{1}{2}\Delta t, x_n + \frac{1}{2}k_1\right),$$
$$k_3 = \Delta t f\left(t_n + \frac{1}{2}\Delta t, x_n + \frac{1}{2}k_2\right), \qquad k_4 = \Delta t f\left(t_n + \Delta t, x_n + k_3\right);$$

$$x_{n+1} = x_n + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right).$$

Example

Solve the initial value problem: $x' = t + 2x$ and $x(0) = 0$

numerically by Euler-method, finding a value for the solution at $t = 1$, and using steps of size $\Delta t = 0.25$.

Solution: $f(t,x)=t+2x$, $x_0=0$ and $t_0=0$.

STEP 1. $t_1= t_0+ \Delta t =0+ 0.25 = 0.25$

$x_1= x_0+ \Delta t\, f(t_0, x_0) =0+ 0.25(0+2*0) = 0$

STEP 2. $t_2= t_1+ \Delta t =0.25+ 0.25 = 0.5$

$x_2= x_1+ \Delta t\, f(t_1, x_1) =0+ 0.25(0.25+2*0) = 0.0625$

STEP 3. $t_3= t_2+ \Delta t =0.5+ 0.25 = 0.75$

$x_3= x_2+ \Delta t\, f(t_2, x_2) =0.0625+ 0.25(0.5+2*0.0625) = 0.21875$

STEP 4. $t_4= t_3+ \Delta t =0.75+ 0.25 = 1$

$x_4= x_3+ \Delta t\, f(t_3, x_3) =0.21875+ 0.25(0.75+2*0.21875) = 0.515625$