

Numerical methods and optimization

Week 1.

Introduction

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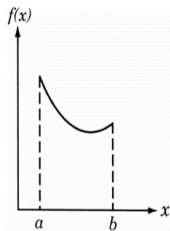
Definition

A function $f \in C[a, b]$ is called *unimodal*, if for some value x^* , it is monotonically increasing for $x \leq x^*$ and monotonically decreasing for $x \geq x^*$, that is the function satisfies the inequality

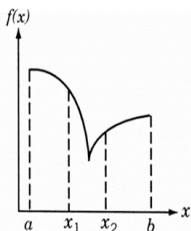
$$x^* < x_2 \Rightarrow f(x_2) > f(x^*)$$

$$x_1 < x^* \Rightarrow f(x_1) > f(x^*)$$

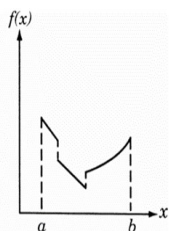
for all $x_1, x_2 \in [a, b]$, $x_1 < x_2$.



(a)



(b)



(c)

We note that the strictly convex and strictly quasi-convex functions are unimodal functions.

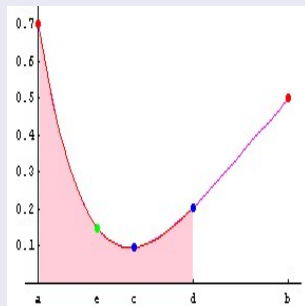
The searching methods of unimodal functions based on the following observations.

Theorem

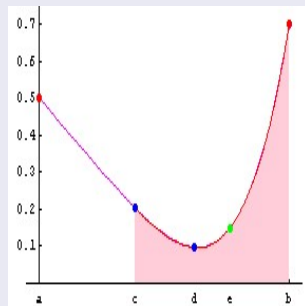
Let $f \in C[a, b]$ be an unimodal function and $a < c < d < b$.

Then

- If $f(c) < f(d)$, then $x^* \in [a, d]$.
- If $f(c) > f(d)$, then $x^* \in [c, b]$.
- If $f(c) = f(d)$, then $x^* \in [c, d]$.



Case 1.



Case 2.

General algorithm of searching methods

```
1  Input  $[a_1, b_1]$ 
2  FOR  $k = 1, 2, \dots$ 
3      Let  $c_k, d_k \in (a_k, b_k)$  and  $c_k < d_k$  .
4      IF  $f(c_k) < f(d_k)$ 
5          THEN  $a_{k+1} = a_k$  ,  $b_{k+1} = d_k$ 
6      IF  $f(c_k) > f(d_k)$ 
7          THEN  $a_{k+1} = c_k$  ,  $b_{k+1} = b_k$ 
8      IF  $f(c_k) = f(d_k)$ 
9          THEN  $a_{k+1} = c_k$  ,  $b_{k+1} = d_k$ 
```

The overlapping sequence of intervals

$$[a_1, b_1] \supset \dots \supset [a_k, b_k] \supset \dots$$

contain the minimum point of the function x^* as a common point. If $b_k - a_k \rightarrow 0$, then we get the approximation of the minimum point (with $\varepsilon > 0$ error) in a finite steps of iteration.

The Dichotomous Search Method

The Dichotomous Search Method computes the midpoint $\frac{a+b}{2}$, and then moves slightly to either side of the midpoint to compute two test points: $\frac{a+b}{2} \pm \delta$, where δ is a very small number. The objective being to place the two test points as close together as possible. The procedure continues until it gets within some small interval containing the optimal solution
Then:

$$c_k = \frac{a_k + b_k}{2} - \delta = a_k + \frac{L_k}{2} - \delta,$$

$$d_k = \frac{a_k + b_k}{2} + \delta = a_k + \frac{L_k}{2} + \delta. \quad k = 1, 2, 3, \dots$$

We can determine the new interval in the following way:

- If $f(c_k) > f(d_k)$, then the new uncertainty interval is $[a_{k+1}, b_{k+1}] = [c_k, b_k]$.
- If $f(c_k) \leq f(d_k)$, then the new uncertainty interval is $[a_{k+1}, b_{k+1}] = [a_k, d_k]$.

It is easy to see, the length of the obtained interval is

$$L_{k+1} = \frac{L_k}{2} + \delta.$$

We continue this procedure until get $L_k < 2\varepsilon$ for some k , where $\varepsilon > 0$ is a tolerance. Then if we choose the midpoint of the last interval $[a_k, b_k]$, then the error will be less than ε .

The algorithm of the Dichotomous Search Method

- 1 Input $[a_1, b_1]$ and $\delta, \varepsilon > 0$
- 2 WHILE $b_k - a_k \geq 2\varepsilon$
- 3 Let $c_k = \frac{a_k + b_k}{2} - \delta = a_k + \frac{L_k}{2} - \delta$,
 and $d_k = \frac{a_k + b_k}{2} + \delta = a_k + \frac{L_k}{2} + \delta$.
- 4 IF $f(c_k) > f(d_k)$
- 5 THEN $a_{k+1} = c_k, b_{k+1} = b_k$
- 6 ELSE $a_{k+1} = a_k, b_{k+1} = c_k$
- 7 $k = k + 1$

The procedure continues until it gets within some small interval containing the optimal solution.

Example

Determine the minimum point of the function

$f(x) = x^2 - 7x + 12 \rightarrow \min!$ with Dichotomous search method,
where $[a, b] = [2, 4]$, $\delta = 0.3$, $\varepsilon = 0.4$.

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Solution

Step 1. The first interval is $[a_1, b_1] = [2, 4]$, which length is $L_1 = 2$. The two test points and values:

$$c_1 = a_1 + \frac{L_1}{2} - \delta = 2.7,$$

$$f(c_1) = 0.39,$$

$$d_1 = a_1 + \frac{L_1}{2} + \delta = 3.3,$$

$$f(d_1) = -0.21$$

Solution

Step 2. Since $f(c_1) > f(d_1)$, then the new uncertainty interval is $[a_2, b_2] = [2.7, 4]$, which length is $L_2 = 1.3$. The two test points and values:

$$\begin{aligned}c_2 &= 3.05, & f(c_2) &= -0.0475, \\d_2 &= 3.65, & f(d_2) &= -0.2275\end{aligned}$$

Step 3. Since $f(c_2) > f(d_2)$, then the new uncertainty interval is $[a_3, b_3] = [2.7, 3.65]$, which length is $L_3 = 0.95$. The two test points and values:

$$\begin{aligned}c_3 &= 2.875, & f(c_3) &= 0.140625, \\d_3 &= 3.475, & f(d_3) &= -0.24937500\end{aligned}$$

Solution

Step 4. Since $f(c_3) > f(d_3)$, then the new uncertainty interval is $[a_4, b_4] = [2.875, 3.65]$, which length is $L_4 = 0.775$. We stop, because the length of obtained interval small enough $L_4 < 2\varepsilon$. The approximation of the minimum point is $x_{\min} \approx 3.2625$. We note the exact minimum point of the function f is $\bar{x} = 3.5$.

The Golden Section Search Method

The Golden Section search method is used to find the maximum or minimum of a unimodal function. To make the discussion of the method simpler, let us assume that we are trying to find the minimum of a function. The previously introduced search method is somewhat inefficient because if the interval is a small number it can take a long time to find the minimum of a function. To improve this efficiency, the Golden Section Search method is suggested. The Golden Section Search Method chooses c_k and d_k such that the one of the two evaluations of the function in each step can be reused in the next step. At the end of the golden section method we get the local minimum point x^* is an interval $[a_k, b_k]$, which length is less than 2ε . We use the approximation $\tilde{x} = \frac{a_k + b_k}{2}$, because the condition $|\tilde{x} - x^*| < \varepsilon$ fulfils.

The algorithm of Golden Section Search Method

Let $\tau = \frac{\sqrt{5}-1}{2} \approx 0.618$.

```

1  INPUT  $[a_1, b_1], \varepsilon > 0, k = 1$ 
2   $c_1 = a_1 + (1 - \tau)(b_1 - a_1), F_c = f(c_1)$ 
3   $d_1 = b_1 - (1 - \tau)(b_1 - a_1), F_d = f(d_1)$ 
4  WHILE  $b_k - a_k \geq 2\varepsilon$ 
5      IF  $F_c < F_d$ 
6          THEN  $a_{k+1} = a_k, b_{k+1} = d_k, d_{k+1} = c_k$ 
7               $c_{k+1} = a_{k+1} + (1 - \tau)(b_{k+1} - a_{k+1})$ 
8               $F_d = F_c, F_c = f(c_{k+1})$ 
9          ELSE  $a_{k+1} = c_k, b_{k+1} = b_k, c_{k+1} = d_k$ 
10              $d_{k+1} = b_{k+1} + (1 - \tau)(b_{k+1} - a_{k+1})$ 
11              $F_c = F_d, F_d = f(d_{k+1})$ 
12          $k = k + 1$ 

```

The length of the uncertainty interval $[a_{k+1}, b_{k+1}]$ is 61.8% of the length of the previous interval of uncertainty $[a_k, b_k]$.

Denote L_k is the length of the interval $[a_k, b_k]$ (i.e.

$L_k = b_k - a_k$). Then we get

$$L_{k+1} = \tau L_k = \tau^2 L_{k-1} = \dots = \tau^k L_1.$$

It means that if we have the first interval $[a_1, b_1]$ and tolerance ε , then we can determine how many steps must be taken to

ensure that the length of the interval is small (that is $L_{k+1} \leq \varepsilon$).

For example, if $L_1 = 1$ and $\varepsilon = 10^{-6}$, then we can solve the

inequality $\tau^k < 2 \cdot 10^{-6}$. We get $k > \frac{\log 2 \cdot 10^{-6}}{\log \tau} \approx 27.2694$,

namely we need 28 steps.

Example

Determine the minimum point of the function $f(x) = x^2 - 7x + 12$ with Golden Section search method, if the interval of uncertainty is $[a, b] = [2, 4]$ and $\varepsilon = 0.3$.

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Determine the minimum point of the function $f(x) = x^2 - 7x + 12$ with Golden Section search method, if the interval of uncertainty is $[a, b] = [2, 4]$ and $\varepsilon = 0.3$.

Solution

Step 1. In the first approximation the uncertainty interval $[a_1, b_1] = [2, 4]$, is of length $L_1 = 2$. The two intermediate points with their corresponding function values:

$$\begin{aligned}c_1 &= a_1 + 0.382L_1 = 2.764, & f(c_1) &= 0.291696, \\d_1 &= a_1 + 0.618L_1 = 3.236, & f(d_1) &= -0.180304\end{aligned}$$

Solution

Step 2 Since $f(c_1) > f(d_1)$, the uncertainty interval $[a_2, b_2] = [2.764, 4]$, is of length $L_2 = 1.236$. The two intermediate points and the associated function values are as follows. Do not forget that now the point c_2 and the point d_1 of the previous interval are identical.

$$\begin{aligned} c_2 &= d_1 = 3.236, & f(c_2) &= -0.180304, \\ d_2 &= a_2 + 0.382L_2 = 3.527848, & f(d_2) &= -0.24922449 \end{aligned}$$

Step 3 Since $f(c_2) > f(d_2)$, the new uncertainty interval $[a_3, b_3] = [3.236, 4]$, is of length $L_3 = 0.764$. The two intermediate points and their associated function values are the following. In this case the point c_3 is taken to be d_2 .

Solution

$$\begin{aligned} c_3 &= d_2 = 3.527848, & f(c_3) &= -0.24922449, \\ d_3 &= a_3 + 0.618L_3 = 3.70806, & f(d_3) &= -0.20671104 \end{aligned}$$

Step 4 Since $f(c_3) < f(d_3)$, the next uncertainty interval $[a_4, b_4] = [3.236, 3.70806]$, is of length $L_4 = 0.47206$. Since $L_4 < 2\varepsilon$ we stop the process. The minimum point is approximated by $x_{\min} \approx 3.47203$.

Now we get closer to the exact minimum point (that is, the $\bar{x} = 3.5$) value as in the case of Dichotomous search.