1 Lagrange Interpolation - 3 Points (Solved via System of Equations)

Example 1.1. *Given the points:*

$$(1, 2), \quad (2, 3), \quad (4, 1)$$

find the interpolating polynomial using the method of solving a system of equations.

Solution 1.2. *We are looking for the quadratic interpolating polynomial in the form:*

$$P(x) = a_0 + a_1 x + a_2 x^2$$

Step 1: Setting up the system of equations

We substitute the given points into the polynomial equation:

$$P(1) = a_0 + a_1 \cdot 1 + a_2 \cdot 1^2 = 2$$
$$P(2) = a_0 + a_1 \cdot 2 + a_2 \cdot 2^2 = 3$$
$$P(4) = a_0 + a_1 \cdot 4 + a_2 \cdot 4^2 = 1$$

This leads to the following system of equations:

$$a_0 + a_1 + a_2 = 2 \tag{1}$$
$$a_0 + 2a_1 + 4a_2 = 3 \tag{2}$$
$$a_0 + 4a_1 + 16a_2 = 1 \tag{3}$$

Step 2: Solving the system of equations

Subtract equation (1) from equation (2):

$$(a_0 + 2a_1 + 4a_2) - (a_0 + a_1 + a_2) = 3 - 2$$
$$a_1 + 3a_2 = 1 \tag{4}$$

Subtract equation (2) from equation (3):

$$(a_0 + 4a_1 + 16a_2) - (a_0 + 2a_1 + 4a_2) = 1 - 3$$
$$2a_1 + 12a_2 = -2$$
$$a_1 + 6a_2 = -1 \tag{5}$$

Now subtract equation (4) from equation (5):

$$(a_1 + 6a_2) - (a_1 + 3a_2) = -1 - 1$$
$$3a_2 = -2 \implies a_2 = -\frac{2}{3}$$

Substitute $a_2 = -\frac{2}{3}$ into equation (4):

$$a_1 + 3\left(-\frac{2}{3}\right) = 1$$
$$a_1 - 2 = 1 \implies a_1 = 3$$

Substitute $a_1 = 3$ and $a_2 = -\frac{2}{3}$ into equation (1):

$$a_0 + 3 + \left(-\frac{2}{3}\right) = 2$$
$$a_0 + \frac{7}{3} = 2 \implies a_0 = 2 - \frac{7}{3} = -\frac{1}{3}$$

Step 3: The interpolating polynomial

The quadratic interpolating polynomial is:

$$\boxed{P(x) = -\frac{1}{3} + 3x - \frac{2}{3}x^2}$$

Step 4: Verification

Check the polynomial at the given points:

$$P(1) = -\frac{1}{3} + 3 \cdot 1 - \frac{2}{3} \cdot 1^2 = -\frac{1}{3} + 3 - \frac{2}{3} = 2$$
$$P(2) = -\frac{1}{3} + 3 \cdot 2 - \frac{2}{3} \cdot 2^2 = -\frac{1}{3} + 6 - \frac{8}{3} = 3$$
$$P(4) = -\frac{1}{3} + 3 \cdot 4 - \frac{2}{3} \cdot 4^2 = -\frac{1}{3} + 12 - \frac{32}{3} = 1$$

Conclusion: *The interpolating polynomial is correct.*

Example 1.3. *Given the points:*

$$(0, 1), \quad (1, 0), \quad (3, 4)$$

find the interpolating polynomial using the method of solving a system of equations.

Solution 1.4. *Assume $P(x) = a_0 + a_1x + a_2x^2$. The system of linear equations*

$$a_0 = 1$$
$$a_0 + a_1 \cdot 1 + a_2 \cdot 1^2 = 0$$
$$a_0 + a_1 \cdot 3 + a_2 \cdot 3^2 = 4$$

Simplify:

$$\begin{aligned} a_0 &= 1 \\ 1 + a_1 + a_2 &= 0 \\ 1 + 3a_1 + 9a_2 &= 4 \end{aligned}$$

Solve:

$$a_2 = 1, \qquad a_1 = -2.$$

Hence

$$\boxed{P(x) = 1 - 2x + x^2}.$$

2 Lagrange Interpolation for 3 Points (Solved via basis functions)

For each i-th point, the Lagrange basis polynomial

$$l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

and

$$P(x) = \sum_{i=0}^{n} f(x_i) l_i(x) = y_0 \cdot l_0(x) + y_1 \cdot l_1(x) + y_2 \cdot l_2(x) + \ldots + y_n \cdot l_n(x)$$

Example 2.1. *Given the points:*

$$(1, 2), \quad (2, 3), \quad (4, 1)$$

find the interpolating polynomial using the Lagrange basic polynomials.

Solution 2.2. *The basis polynomials:*

$$\begin{aligned} l_0(x) &= \frac{(x-2)(x-4)}{(1-2)(1-4)} = \frac{(x-2)(x-4)}{3} \\ l_1(x) &= \frac{(x-1)(x-4)}{(2-1)(2-4)} = -\frac{(x-1)(x-4)}{2} \\ l_2(x) &= \frac{(x-1)(x-2)}{(4-1)(4-2)} = \frac{(x-1)(x-2)}{6} \end{aligned}$$

Interpolating polynomial:

$$\begin{aligned} P(x) &= 2l_0(x) + 3l_1(x) + 1l_2(x) \\ &= 2\frac{(x-2)(x-4)}{3} + 3\left(-\frac{(x-1)(x-4)}{2}\right) + \frac{(x-1)(x-2)}{6} \end{aligned}$$

Simplify to get:

$$\boxed{P(x) = -\frac{1}{3} + 3x - \frac{2}{3}x^2}.$$

Example 2.3. *Given the points:*

$$(0,1), \quad (1,0), \quad (3,4)$$

find the interpolating polynomial using the Lagrange basic polynomials.

Solution 2.4. *Basis polynomials:*

$$\begin{aligned} l_0(x) &= \frac{(x-1)(x-3)}{(0-1)(0-3)} = \frac{(x-1)(x-3)}{3} \\ l_1(x) &= \frac{(x-0)(x-3)}{(1-0)(1-3)} = -\frac{x(x-3)}{2} \\ l_2(x) &= \frac{(x-0)(x-1)}{(3-0)(3-1)} = \frac{x(x-1)}{6} \end{aligned}$$

Interpolating polynomial:

$$P(x) = 1 \cdot l_0(x) + 0 \cdot l_1(x) + 4 \cdot l_2(x)$$

Simplify:

$$P(x) = 1 \cdot \frac{(x-1)(x-3)}{3} + 4 \cdot \frac{x(x-1)}{6} = 1 - 2x + x^2$$

Example 2.5. *Given the points:*

x	0	1	2	3
y	1	3	2	4

find the interpolating polynomial using the Lagrange basic polynomials.

Solution 2.6. *Basis polynomials $l_i(x)$ are:*

$$
\begin{aligned}
l_0(x) &= \frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)} = -\frac{(x-1)(x-2)(x-3)}{6} \\
l_1(x) &= \frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)} = \frac{x(x-2)(x-3)}{2} \\
l_2(x) &= \frac{(x-0)(x-1)(x-3)}{(2-0)(2-1)(2-3)} = -\frac{x(x-1)(x-3)}{2} \\
l_3(x) &= \frac{(x-0)(x-1)(x-2)}{(3-0)(3-1)(3-2)} = \frac{x(x-1)(x-2)}{6}
\end{aligned}
$$

The Lagrange interpolating polynomial:

$$
\boxed{P(x) = 1 \cdot l_0(x) + 3 \cdot l_1(x) + 2 \cdot l_2(x) + 4 \cdot l_3(x)}
$$

Example 2.7. *Given points:*

$$
(0,1), \quad (1,3), \quad (3,-2), \quad (4,5)
$$

Construct the interpolating polynomial $P(x)$ using Lagrange basis polynomials!

Solution 2.8. ***Step 1: Lagrange basis polynomials***

$$
\begin{aligned}
l_0(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{(x-1)(x-3)(x-4)}{(0-1)(0-3)(0-4)} \\
&= \frac{(x-1)(x-3)(x-4)}{(-1)(-3)(-4)} = \frac{(x-1)(x-3)(x-4)}{-12} \\
l_1(x) &= \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{(x-0)(x-3)(x-4)}{(1-0)(1-3)(1-4)} \\
&= \frac{x(x-3)(x-4)}{(1)(-2)(-3)} = \frac{x(x-3)(x-4)}{6} \\
l_2(x) &= \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{(x-0)(x-1)(x-4)}{(3-0)(3-1)(3-4)} \\
&= \frac{x(x-1)(x-4)}{(3)(2)(-1)} = \frac{x(x-1)(x-4)}{-6} \\
l_3(x) &= \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{(x-0)(x-1)(x-3)}{(4-0)(4-1)(4-3)} \\
&= \frac{x(x-1)(x-3)}{(4)(3)(1)} = \frac{x(x-1)(x-3)}{12}
\end{aligned}
$$

Step 2: Write the Interpolating Polynomial *$P(x)$ The interpolating polynomial is:*

$$P(x) = y_0 \cdot l_0(x) + y_1 \cdot l_1(x) + y_2 \cdot l_2(x) + y_3 \cdot l_3(x)$$

Substitute the y_i values:

$$\begin{aligned} P(x) &= 1 \cdot l_0(x) + 3 \cdot l_1(x) + (-2) \cdot l_2(x) + 5 \cdot l_3(x) \\ &= \frac{(x-1)(x-3)(x-4)}{-12} + 3 \cdot \frac{x(x-3)(x-4)}{6} \\ &\quad + (-2) \cdot \frac{x(x-1)(x-4)}{-6} + 5 \cdot \frac{x(x-1)(x-3)}{12} \end{aligned}$$

Simplify the coefficients:

$$\begin{aligned} P(x) &= \frac{(x-1)(x-3)(x-4)}{-12} + \frac{3}{6} \cdot x(x-3)(x-4) \\ &\quad + \frac{2}{6} \cdot x(x-1)(x-4) + \frac{5}{12} \cdot x(x-1)(x-3) \\ &= \frac{(x-1)(x-3)(x-4)}{-12} + \frac{1}{2} \cdot x(x-3)(x-4) \\ &\quad + \frac{1}{3} \cdot x(x-1)(x-4) + \frac{5}{12} \cdot x(x-1)(x-3) \end{aligned}$$

Step 3: Expand Each Term

the first term:

$$\begin{aligned} (x-1)(x-3)(x-4) &= (x-1)[(x-3)(x-4)] \\ &= (x-1)(x^2-7x+12) \\ &= x(x^2-7x+12) - 1(x^2-7x+12) \\ &= x^3 - 7x^2 + 12x - x^2 + 7x - 12 \\ &= x^3 - 8x^2 + 19x - 12 \end{aligned}$$

Then:

$$\frac{x^3-8x^2+19x-12}{-12} = -\frac{1}{12}x^3 + \frac{2}{3}x^2 - \frac{19}{12}x + 1$$

The second term:

$$\begin{aligned} x(x-3)(x-4) &= x[(x-3)(x-4)] = x(x^2-7x+12) \\ &= x^3 - 7x^2 + 12x \end{aligned}$$

Then:

$$\frac{1}{2}(x^3 - 7x^2 + 12x) = \frac{1}{2}x^3 - \frac{7}{2}x^2 + 6x$$

The third term:

$$\begin{aligned} x(x-1)(x-4) &= x[(x-1)(x-4)] = x(x^2-5x+4) \\ &= x^3 - 5x^2 + 4x \end{aligned}$$

Then:

$$\frac{1}{3}(x^3 - 5x^2 + 4x) = \frac{1}{3}x^3 - \frac{5}{3}x^2 + \frac{4}{3}x$$

Finally the fourth term:

$$\begin{aligned} x(x-1)(x-3) &= x[(x-1)(x-3)] = x(x^2-4x+3) \\ &= x^3 - 4x^2 + 3x \end{aligned}$$

Then:

$$\frac{5}{12}(x^3 - 4x^2 + 3x) = \frac{5}{12}x^3 - \frac{20}{12}x^2 + \frac{15}{12}x = \frac{5}{12}x^3 - \frac{5}{3}x^2 + \frac{5}{4}x$$

Step 4: Sum All Terms

The cubic terms x^3:

$$-\frac{1}{12} + \frac{1}{2} + \frac{1}{3} + \frac{5}{12} = \frac{-1}{12} + \frac{6}{12} + \frac{4}{12} + \frac{5}{12} = \frac{14}{12} = \frac{7}{6}$$

The quadratic terms x^2:

$$\frac{2}{3} - \frac{7}{2} - \frac{5}{3} - \frac{5}{3} = \frac{4}{6} - \frac{21}{6} - \frac{10}{6} - \frac{10}{6} = \frac{-37}{6}$$

The linear terms x:

$$-\frac{19}{12} + 6 + \frac{4}{3} + \frac{5}{4} = -\frac{19}{12} + \frac{72}{12} + \frac{16}{12} + \frac{15}{12} = \frac{84}{12} = 7$$

Finally the constant term is 1.

Step 5: Final Answer

$$\boxed{P(x) = \frac{7}{6}x^3 - \frac{37}{6}x^2 + 7x + 1}$$

Newton Interpolation (with divided differences)

Example 2.9. *Given the following points:*

$$(1, 2), \quad (2, 3), \quad (4, 5)$$

Find the Newton interpolating polynomial $P(x)$.

Solution 2.10. ***Step 1: Create the Divided Difference Table***

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$
1	*2*		
2	*3*		
4	*5*		

Step 2: Compute First-Order Divided Differences

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{3-2}{2-1} = 1$$
$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1} = \frac{5-3}{4-2} = 1$$

Step 3: Compute Second-Order Divided Difference

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{1-1}{4-1} = 0$$

Step 4: Fill in the Divided Difference Table

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$
1	*2*	*1*	*0*
2	*3*	*1*	
4	*5*		

Step 5: Write the Newton Interpolating Polynomial

The Newton interpolation formula is:

$$P(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

Substitute the values:

$$\begin{aligned} P(x) &= 2 + 1 \cdot (x-1) + 0 \cdot (x-1)(x-2) \\ &= 2 + (x-1) = x + 1 \end{aligned}$$

Final Answer

$$\boxed{P(x) = x + 1}$$

Example 2.11. *Given the following points:*

$$(1, 2), \quad (2, 5), \quad (4, 17)$$

Find the Newton interpolating polynomial $P(x)$.

Solution 2.12. *The divided differences table:*

$$f[x_0] = 2$$
$$f[x_1] = 5$$
$$f[x_2] = 17$$
$$f[x_0, x_1] = \frac{5-2}{2-1} = 3$$
$$f[x_1, x_2] = \frac{17-5}{4-2} = 6$$
$$f[x_0, x_1, x_2] = \frac{6-3}{4-1} = 1$$

and the interpolating polynomial:

$$P(x) = 2 + 3(x-1) + 1(x-1)(x-2) = 2 + 3x - 3 + x^2 - 3x - 2 = x^2 + 1$$

Example 2.13. *Given the following data points:*

$$(0, 3), \quad (1, 6), \quad (2, 11), \quad (3, 18)$$

Find the Newton interpolating polynomial $P(x)$ using divided differences.

Solution 2.14. ***Step 1: Create the Divided Difference Table Structure***

We set up the table for x_i, $f[x_i]$, and their divided differences.

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
0	*3*			
1	*6*			
2	*11*			
3	*18*			

Step 2: Compute First-Order Divided Differences

$$f[x_0, x_1] = \frac{6-3}{1-0} = 3$$
$$f[x_1, x_2] = \frac{11-6}{2-1} = 5$$
$$f[x_2, x_3] = \frac{18-11}{3-2} = 7$$

Step 3: Compute Second-Order Divided Differences

$$f[x_0, x_1, x_2] = \frac{5-3}{2-0} = \frac{2}{2} = 1$$
$$f[x_1, x_2, x_3] = \frac{7-5}{3-1} = \frac{2}{2} = 1$$

Step 4: Compute Third-Order Divided Difference

$$f[x_0, x_1, x_2, x_3] = \frac{1-1}{3-0} = \frac{0}{3} = 0$$

Step 5: Fill in the Complete Divided Difference Table

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
0	*3*	*3*	*1*	*0*
1	*6*	*5*	*1*	
2	*11*	*7*		
3	*18*			

Step 6: Write the Newton Interpolating Polynomial

The Newton interpolation formula is:

$$\begin{aligned} P(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\ + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2) \end{aligned}$$

Substitute the values:

$$\begin{aligned} P(x) &= 3 + 3(x - 0) + 1(x - 0)(x - 1) + 0(x - 0)(x - 1)(x - 2) \\ &= 3 + 3x + x^2 - x = x^2 + 2x + 3 \end{aligned}$$

Final Answer

$$\boxed{P(x) = x^2 + 2x + 3}$$

Check

We can verify that

$$\begin{aligned} P(0) &= 0^2 + 2 \cdot 0 + 3 = 3, \\ P(1) &= 1^2 + 2 \cdot 1 + 3 = 6, \\ P(2) &= 2^2 + 2 \cdot 2 + 3 = 11, \\ P(3) &= 3^2 + 2 \cdot 3 + 3 = 18. \end{aligned}$$

The interpolating polynomial correctly passes through all given points.

Example 2.15. *Given the following data:*

- $f(1) = 2$
- $f'(1) = 3$

- $f(2) = 3$

Find the Hermite interpolating polynomial $P(x)$.

Solution 2.16. ***Step 1:*** *Since we have both $f(1)$ and $f'(1)$, we will repeat the point $x = 1$. The data points become:*

$$x_0 = 1, \quad x_1 = 1, \quad x_2 = 2$$

and

$$\begin{aligned} f[x_0] &= f(1) = 2 \\ f[x_1] &= f(1) = 2 \\ f[x_2] &= f(2) = 3 \end{aligned}$$

Step 2: Compute divided differences

- *The First (First-Order) divided difference:*

  $$f[x_0, x_1] = f'(1) = 3$$

  (By definition for repeated nodes)

- *Next (First-Order) divided difference:*

  $$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1} = \frac{3 - 2}{2 - 1} = 1$$

- *The Second-Order divided difference:*

  $$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{1 - 3}{2 - 1} = -2$$

Step 3: Build the Hermite Interpolation Polynomial
The Hermite interpolation polynomial is:

$$P(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

Substitute the values:

$$\begin{aligned} P(x) &= 2 + 3(x - 1) - 2(x - 1)(x - 1) = 2 + 3(x - 1) - 2(x - 1)^2 \\ &= 2 + 3x - 3 - 2(x^2 - 2x + 1) = -2x^2 + 7x - 3 \end{aligned}$$

Final Answer:

$$\boxed{P(x) = -2x^2 + 7x - 3}$$

Check: *First we compute*

$$P'(x) = -4x + 7,$$

and

$$\begin{aligned} P(1) &= -2 \cdot 1^2 + 7 \cdot 1 - 3 = -2 + 7 - 3 = 2, \\ P'(1) &= -4 \cdot 1 + 7 = 3 \\ P(2) &- 2 \cdot 2^2 + 7 \cdot 2 - 3 = -8 + 14 - 3 = 3. \end{aligned}$$

Correct!

Example 2.17. *Given the points:*

$$\begin{aligned} x_0 &= 1, \qquad f(1) = 2, \qquad f'(1) = 3 \\ x_1 &= 2, \qquad f(2) = 5, \qquad f'(2) = 7 \end{aligned}$$

Find the Hermite interpolating polynomial $P(x)$.

Solution 2.18. *First we construct the divided differences table.*

Node	*f or divided difference*
1	2
1	$f[1,1] = 3$
2	$f[1,1,2] = \frac{5-2}{2-1} = 3$
2	$f[1,1,2,2] = \frac{7-3}{2-1} = 4$

The interpolating polynomial:

$$\begin{aligned} P(x) &= f[1] + f[1,1](x-1) + f[1,1,2](x-1)^2 + f[1,1,2,2](x-1)^2(x-2) \\ &= 2 + 3(x-1) + 3(x-1)^2 + 4(x-1)^2(x-2) \end{aligned}$$

3 Spline interpolation

Example 3.1. *Given the points:*

$$(0,1), \quad (1,3), \quad (2,2)$$

Construct a linear Spline interpolating these points.

Solution 3.2. *We have two intervals. The first is $[0,1]$. The linear spline*

$$S_1(x) = 1 + \frac{3-1}{1-0}(x-0) = 1 + 2x$$

The second interval is $[1, 2]$*, and the linear spline is*

$$S_2(x) = 3 + \frac{2-3}{2-1}(x-1) = 3 - (x-1) = 4 - x$$

Final Answer

$$\boxed{S(x) = \begin{cases} S_1(x) = 1 + 2x & \textit{for } x \in [0,1] \\ \\ S_2(x) = 4 - x & \textit{for } x \in [1,2] \end{cases}}$$

Example 3.3. *Given the points:*

$$(0,1), \quad (1,2), \quad (3,2)$$

Construct a spline interpolating these points, which is linear in the first interval and quadratic in the second interval.

Solution 3.4. *Assume:* $S_1(x)$ *is linear on the interval* $[0, 1]$*:*

$$S_1(x) = a_1 x + b_1$$

and $S_2(x)$ *is quadratic on the interval* $[1, 3]$*:*

$$S_2(x) = a_2 x^2 + b_2 x + c_2$$

Conditions:

- $S_1(0) = 1 \rightarrow b_1 = 1$
- $S_1(1) = 2 \rightarrow a_1 + 1 = 2 \rightarrow a_1 = 1$
- $S_2(1) = 2 \rightarrow a_2 \cdot 1^2 + b_2 \cdot 1 + c_2 = 2$
- $S_2(3) = 2 \rightarrow a_2 \cdot 3^2 + b_2 \cdot 3 + c_2 = 2$
- $S_1'(1) = S_2'(1) \rightarrow a_1 = 2a_2 \cdot 1 + b_2$

Solution:

$$a_1 = 1 \rightarrow 2a_2 + b_2 = 1$$
$$\textit{From } S_2(1) = 2 \rightarrow a_2 + b_2 + c_2 = 2$$
$$\textit{From } S_2(3) = 2 \rightarrow 9a_2 + 3b_2 + c_2 = 2$$

Solution of this system is

$$a_2 = -0.5, \quad b_2 = 2, \quad c_2 = 0.5$$

which implies

$$S_1(x) = x + 1$$

and

$$S_2(x) = -0.5x^2 + 2x + 0.5$$

Final Answer

$$\boxed{S(x) = \begin{cases} S_1(x) = x + 1 & \textit{for } x \in [0, 1] \\ \\ S_2(x) = -0.5x^2 + 2x + 0.5 & \textit{for } x \in [1, 3] \end{cases}}$$

3.1 Natural cubic Spline interpolation

Example 3.5. *Given the points:*

$$(1, 2), \quad (2, 3), \quad (3, 5)$$

Construct the natural cubic spline interpolating these points.

Solution 3.6. *We will define two spline segments:*

$$S(x) = \begin{cases} S_1(x) & x \in [1, 2] \\ S_2(x) & x \in [2, 3] \end{cases}$$

Each cubic spline has the form:

$$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3,$$

that is

$$S(x) = \begin{cases} a_1 + b_1(x - 1) + c_1(x - 1)^2 + d_3(x - 1)^3, & x \in [1, 2] \\ a_2 + b_2(x - 2) + c_2(x - 2)^2 + d_3(x - 2)^3, & x \in [2, 3] \end{cases}$$

***Step 1: Conditions for** $S(x)$*

1. *Interpolation conditions:*

$$S_1(1) = 2, \quad S_1(2) = 3.$$

   and

$$S_2(2) = 3, \quad S_2(3) = 5.$$

2. *Continuity of first derivative at $x = 2$:*

$$S_1'(2) = S_2'(2).$$

3. *Continuity of second derivative at $x = 2$:*

$$S_1''(2) = S_2''(2).$$

4. *Natural spline boundary conditions:*

$$S_1''(1) = 0, \quad S_2''(3) = 0.$$

Step 3: Apply the Conditions.
1.) The interpolation Conditions:
For $S_1(x) = a_1 + b_1(x-1) + c_1(x-1)^2 + d_1(x-1)^3$.

$$\begin{aligned} &S_1(1) = a_1 = 2, \\ &S_1(2) = 2 + b_1(1) + c_1(1)^2 + d_1(1)^3 = 3 \Rightarrow b_1 + c_1 + d_1 = 1 \end{aligned}$$

and for $S_2(x) = a_2 + b_2(x-2) + c_2(x-2)^2 + d_2(x-2)^3$

$$\begin{aligned} &S_2(2) = a_2 = 3, \\ &S_2(3) = 3 + b_2(1) + c_2(1)^2 + d_2(1)^3 = 5 \qquad \Rightarrow b_2 + c_2 + d_2 = 2. \end{aligned}$$

2.) Continuity of First Derivatives at $x = 2$: Hence

$$\begin{aligned} S_1'(x) &= b_1 + 2c_1(x-1) + 3d_1(x-1)^2, \\ S_2'(x) &= b_2 + 2c_2(x-2) + 3d_2(x-2)^2. \end{aligned}$$

Therefore

$$\begin{aligned} S_1'(2) &= b_1 + 2c_1(1) + 3d_1(1)^2 = b_1 + 2c_1 + 3d_1, \\ S_2'(2) &= b_2 \qquad \Rightarrow b_1 + 2c_1 + 3d_1 = b_2. \end{aligned}$$

3.) Continuity of Second Derivatives at $x = 2$: First, we compute the second order derivatives

$$\begin{aligned} S_1''(x) &= 2c_1 + 6d_1(x-1), \\ S_2''(x) &= 2c_2 + 6d_2(x-2) \end{aligned}$$

and we get

$$\begin{aligned} S_1''(2) &= 2c_1 + 6d_1(1) = 2c_1 + 6d_1, \\ S_2''(2) &= 2c_2 \qquad \Rightarrow 2c_1 + 6d_1 = 2c_2. \end{aligned}$$

4.) Natural Spline Boundary Conditions:

$$S_1''(1) = 2c_1 + 6d_1(0) = 2c_1 = 0 \qquad\qquad \Rightarrow c_1 = 0.$$

and

$$S_2''(3) = 2c_2 + 6d_2(1) = 2c_2 + 6d_2 = 0.$$

Step 4: Solve the System of Equations

From $c_1 = 0$:

The first condition $b_1 + c_1 + d_1 = 1$ becomes:

$$b_1 + d_1 = 1.$$

From derivative continuity:

$$b_1 + 2 \cdot 0 + 3d_1 = b_2 \Rightarrow b_1 + 3d_1 = b_2.$$

From second derivative continuity:

$$2 \cdot 0 + 6d_1 = 2c_2 \Rightarrow 6d_1 = 2c_2 \Rightarrow c_2 = 3d_1.$$

From the condition:

$$b_2 + c_2 + d_2 = 2.$$

Substitute $c_2 = 3d_1$:

$$b_2 + 3d_1 + d_2 = 2.$$

From natural condition on $S_2''(3)$:

$$2c_2 + 6d_2 = 0$$

and

$$2 \cdot (3d_1) + 6d_2 = 0 \Rightarrow 6d_1 + 6d_2 = 0 \Rightarrow d_2 = -d_1$$

Now solve:

$$b_1 + d_1 = 1 \Rightarrow b_1 = 1 - d_1$$

and

$$b_2 = b_1 + 3d_1 = (1 - d_1) + 3d_1 = 1 + 2d_1.$$

Substitute into

$$\begin{aligned} b_2 + 3d_1 + d_2 &= 2 \\ (1 + 2d_1) + 3d_1 + (-d_1) &= 2 \\ 1 + 2d_1 + 3d_1 - d_1 &= 2 \\ 1 + 4d_1 &= 2 \Rightarrow d_1 = \frac{1}{4} \end{aligned}$$

Then:

$$
\begin{aligned}
b_1 &= 1 - \frac{1}{4} = \frac{3}{4} \\
c_1 &= 0 \\
d_1 &= \frac{1}{4} \\
b_2 &= 1 + 2 \cdot \frac{1}{4} = 1 + \frac{1}{2} = \frac{3}{2} \\
c_2 &= 3 \cdot \frac{1}{4} = \frac{3}{4} \\
d_2 &= -d_1 = -\frac{1}{4}
\end{aligned}
$$

Step 5: Write the Spline Functions

For $x \in [1, 2]$:

$$
\begin{aligned}
S_1(x) &= 2 + \frac{3}{4}(x-1) + 0 + \frac{1}{4}(x-1)^3 = 2 + \frac{3x}{4} - \frac{3}{4} + \frac{x^3}{4} - \frac{3x^2}{4} + \frac{3x}{4} - \frac{1}{4} \\
&= \frac{x^3}{4} - \frac{3x^2}{4} + \frac{3x}{2} + 1.
\end{aligned}
$$

For $x \in [2, 3]$:

$$
\begin{aligned}
S_2(x) &= 3 + \frac{3}{2}(x-2) + \frac{3}{4}(x-2)^2 - \frac{1}{4}(x-2)^3 \\
&= 3 + \frac{3x}{2} - 3 + \frac{3x^2}{4} - 3x + 3 - \frac{x^3}{4} + \frac{3x^2}{2} - 3x + 2 \\
&= -\frac{x^3}{4} + \frac{9x^2}{4} - \frac{9x}{2} + 5.
\end{aligned}
$$

Final Answer

$$
\boxed{S(x) = \begin{cases} S_1(x) = \dfrac{x^3}{4} - \dfrac{3x^2}{4} + \dfrac{3x}{2} + 1 & \textit{for } x \in [1, 2] \\[2ex] S_2(x) = -\dfrac{x^3}{4} + \dfrac{9x^2}{4} - \dfrac{9x}{2} + 5 & \textit{for } x \in [2, 3] \end{cases}}
$$