1 Basic Concepts of Optimization

Optimization is a methodology aiming to find the best among available alternatives. The available alternatives are referred to as *feasible solutions*, and their quality is measured using some numerical function called the *objective function.* A feasible solution that yields the best (minimum or maximum) objective function value is called an *optimal solution.*

Steps of formulating an optimization model:

1. Introduce the *decision variables*, which are the parameters whose values need to be determined in order to solve the problem.

2. State the *objective function*, i.e., the quantity that we need to optimize as a function of the decision variables.

3. Specify the *constraints*, i.e., the conditions that the design parameters are required to satisfy.

Mathematical description: Let $X \subseteq \mathbb{R}^n$ be a set of n-dimensional real vectors in the form $x = [x_1, \dots, x_n]^T$, and let $f : X \to \mathbb{R}$ be a given function. In mathematical terms, an optimization problem has the general form

$$\begin{array}{rcr} \text{maximize } f(x) & \quad\text{or}\quad & \text{minimize } f(x) \\ \text{subject to } x \in X & & \text{subject to } x \in X. \end{array}$$

In the above, each x_j, $j = 1, 2 \dots n$, is called a *decision variable*, X is called the *feasible (admissible) region*, and f is the *objective function.*

Note that a maximization problem $\max_{x \in X} f(x)$ can be easily converted into an equivalent minimization problem $\min_{x \in X} (-f(x))$ and vice versa.

We will typically use a *functional form* of an optimization problem, in which the feasible region is described by a set of *constraints* given by equalities and inequalities in the form

$$\begin{aligned} h_i(x) &= 0, \quad i = 1, 2, \dots p, \\ g_j(x) &\leq 0, \quad j = 1, 2, \dots r, \end{aligned}$$

where $h_i : \mathbb{R}^n \to \mathbb{R},\ i = 1, 2, \dots p,\ g_j : \mathbb{R}^n \to \mathbb{R},\ j = 1, 2, \dots r$ are given functions.

Definition 1 (Global Minimum)**.** The point $x^* \in X$ is a point of global minimum (global minimizer) for the problem $\min_{x \in X} f(x)$ if

$$f(x^*) \leq f(x) \text{ for all } x \in X.$$

Definition 2 (Local Minimum)**.** The point $x^* \in X$ is a point of local minimum (local minimizer) for the problem $\min_{x \in X} f(x)$ if there exists $\epsilon > 0$ such that

$$f(x^*) \leq f(x) \text{ for any } x \in X \text{ with } ||x - x^*|| \leq \epsilon.$$

Remark 1. The concepts of global and local maximizers can be easily given by writing

$$f(x^*) \geq f(x)$$

instead of $f(x^*) \leq f(x)$ in the definition.

Classification of optimization models: We can group the models

- by the domain of the decision variables:
  - continuous models: the variables can be any points of an interval
  - discrete models: the variables can be only determined numbers, e.g. integers.
- by the functions of the model
  - linear model: all constraints and the objective function are linear
  - nonlinear model: at least one of the functions is nonlinear
- by the dependence of randomness
  - deterministic model: all the parameters are constant
  - stochastic model: some parameters are random variables

2 Solving Linear Programming (LP) Problems

Linear programming is a methodology for solving linear optimization problems, in which one wants to optimize a linear objective function subject to constraints on its variables expressed in terms of linear equalities and/or inequalities. Ever since the introduction of the simplex method by George Dantzig in the late 1940s, linear programming has played a major role in shaping the modern horizons of the field of optimization and its applications.

2.1 Solving two-variable LP Problems Graphically

In linear programming problems the objective function and all the constraints are linear. In this chapter we study simple LP problems with two decision variables, x_1 and x_2. In this case we can solve the problem graphically after plotting the set of feasible points in two-dimensional space. Instead of x_1 and x_2 we denote the variables by x and y as usual in coordinate geometry.

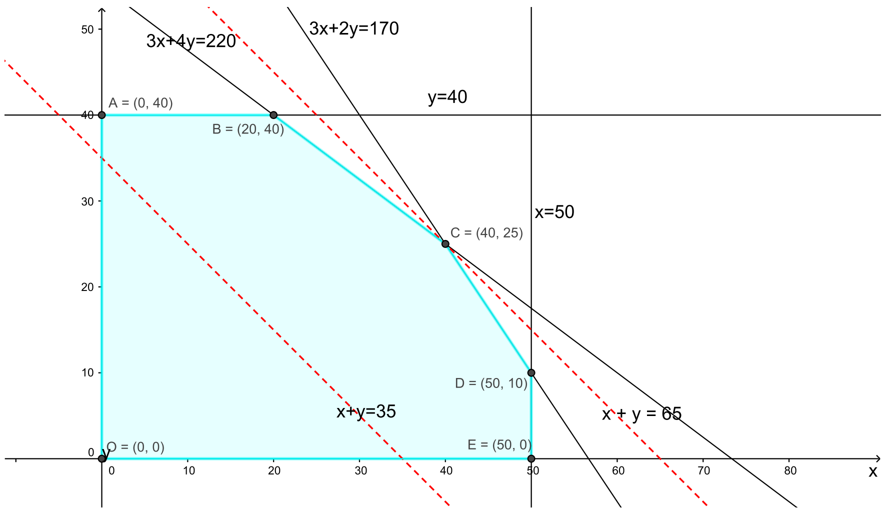
2.1.1 Sandwich problem (a maximization problem)

A mother makes two types of sandwiches for the birthday party of her son. A salami sandwich consists of 3 g butter, 3 slices of egg and two slices of salami. A ham sandwich consists of 4 g butter, 2 slices of egg and a slice of ham. (Of course these ingredients are in between two slices of bread in both cases.) For making sandwiches 100 slices of salami, 40 slices of ham, 170 slices of egg and 220 g butter are available. There is plenty of bread, so no restriction is needed for bread. How many sandwiches of each type should be made to maximize the total number of sandwiches?

Solution: First we have to set the mathematical model of the problem, which means that after introducing suitable variables we give the objective function and formulate the constraints. Suppose that the number of salami sandwiches to be made is x, and the number of ham sandwiches is y. A salami sandwich requires $3x$ g butter, a ham sandwich requires $4y$ g butter, but only 220 g is available, so the inequality $3x + 4y \leq 220$ must be valid. Similarly, we need altogether $3x + 2y$ slices of egg, which amount can be at most 170. We have the constraint $2x \leq 100$ for the salami, and $y \leq 40$ in the case of ham. In addition, neither x or y can be negative, that is, $x \geq 0$ and $y \geq 0$. We look for the maximal value of the amount $x + y$ subject to the previous constraints. Summarizing, the model can be formulated as follows:

$$
\begin{aligned}
x + y &\longrightarrow \max \\
3x + 4y &\leq 220 \\
3x + 2y &\leq 170 \\
2x &\leq 100 \\
y &\leq 40 \\
x &\geq 0 \\
y &\geq 0
\end{aligned}
$$

We solve the problem graphically, plotting the points satisfying all the six constraints. For example points $(x; y)$ for which $3x+4y \leq 220$ are in the half-plane under the line $3x+4y = 220$, and the points satisfying the inequality $y \geq 0$ are above the x axis. The shadowed region in the next figure illustrates the set L of the feasible solutions, its points satisfy all constraints. We have to select the point in L which corresponds to the maximal objective function value: for which the sum of its coordinates is maximal. We fix a number z and consider the line $x+y = z$. (The line where the objective function is constant is often called the *isoprofit line.*) We need the maximal z, for which the line $x+y = z$ and the polygon L have at least one common point. The value z can be read from the y axis, so we have to move a line with slope -1 parallel to itself and over the set L, and find where it intersects the y axis at the highest point. In our case we could push the line to the point (40;25) of L, z is maximal in this point and its value is 65. The solution of the problem: 40 salami sandwiches and 25 ham sandwiches can be made for the party, and the optimum is 65.



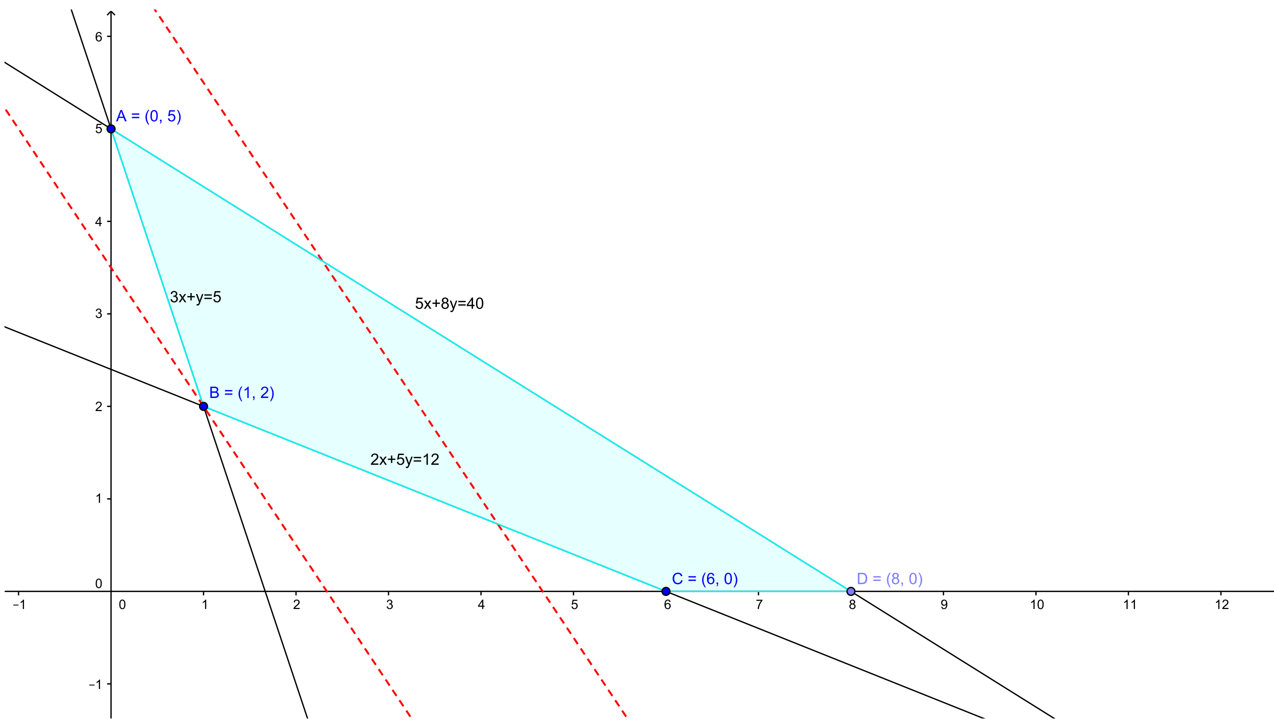
2.1.2 Diet problem (a minimization problem)

A person is on a diet and eats two types of fruit, lemon and apple, to cover his vitamin C and iron needs. Suppose that the minimum recommended daily intake is 500 units of vitamin C, and 1.2 mg from iron. 100 g lemon contains 300 units of vitamin C and 0.2 mg iron, and there are 100 units of vitamin C and 0.5 mg iron in 100 g apple. The person wants to take in at most 400 calories by eating the fruits. 100 g lemon provides 50 calories and 100 g apple provides 80 calories. At the grocery's shop one kg lemon costs 600 Ft and one kg apple is 400 Ft. What combination of fruit should be bought to cover the needs, not to exceed the calories and to minimize the price of the fruit?

Solution: Suppose that the person buys $x \cdot 100$ g lemon and $y \cdot 100$ g apple in the shop. The price is $60x + 40y$,it is to be minimized. Both variables are nonnegative and we have three further constraints:

$$
\begin{aligned}
60x + 40y &\longrightarrow \min \\
300x + 100y &\geq 500 \\
0.2x + 0.5y &\geq 1.2 \\
50x + 80y &\leq 400 \\
x &\geq 0 \\
y &\geq 0
\end{aligned}
$$

We plot the set of the feasible solutions in two-dimensional space. Similarly to the sandwich problem, we have again a polygon in the plane (see figure). If we fix the value of the objective function in z, and express variable y, the equation $y = -\frac{3}{2}x + z$ is obtained. In this case we have to move a line with slope -1.5 over the polygon, and seek the lowest point, where the line intersects the y axis. This line leaves the polygon at point (1;2) and intersects the y axis at 3.5. This means that the person needs 100 g lemon and 200 g apple a day, and it costs 140 Ft. (Some explanation to the optimum: we obtain 140 by multiplying the value 3.5 by 40, because in the beginning, the objective function was divided by 40.)



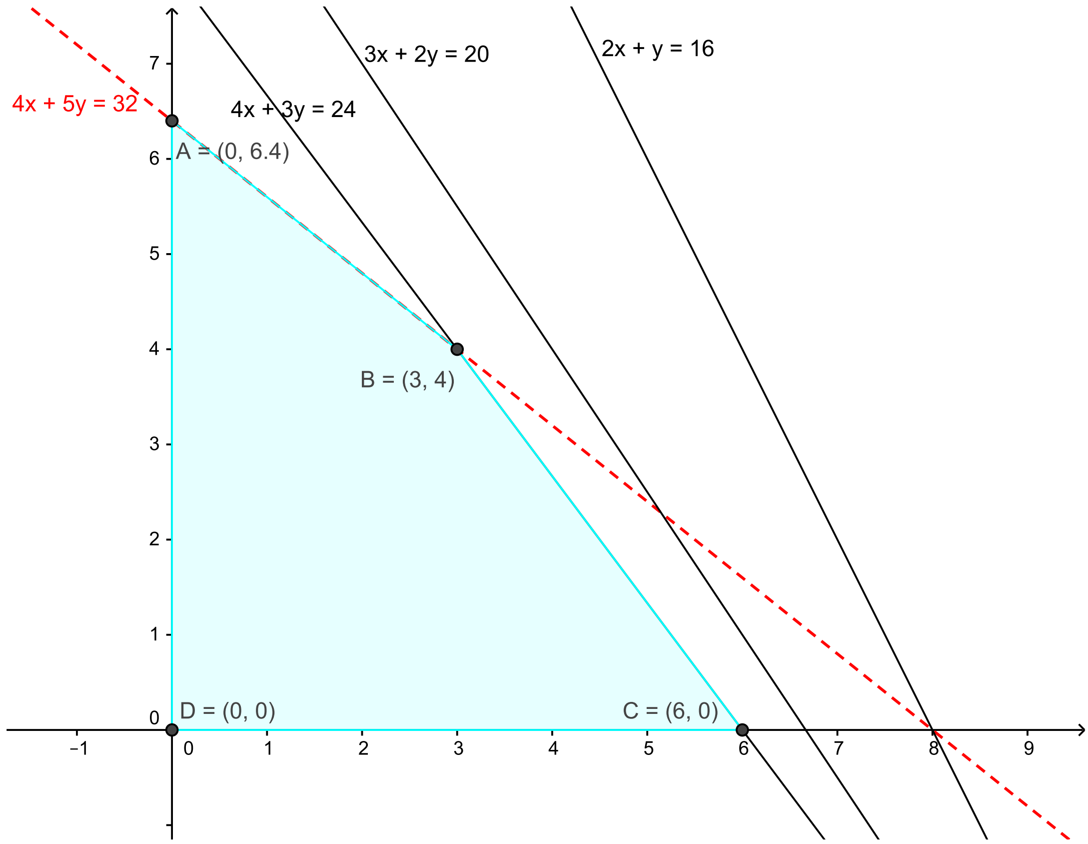
2.1.3 A problem with multiple optimal solutions

A builder finds he is commonly asked to build two types of buildings, A and B. The profits per building are \$4,000 and \$5,000 for A and B, respectively. There are some restrictions on available materials. Building A requires 4,000 m^2 of timber, 4 t of steel, 300 m^2 of roofing iron and 200 $m^3$of concrete. Building B requires 5000 m^2 timber, 3 t of steel, 200 m^2 of roofing iron and 100 m^3 of concrete. However only 32,000 m^2 of timber, 24 t of steel, 2,000 m^2 of roofing iron and 1600 m^3 of concrete are available per year. What combination of A and B should be built per year to maximize profit?

Solution: Let x the number of building A, and y the number of building B to be constructed. The problem can now be modeled as follows:

$$
\begin{aligned}
4000x + 5000y &\longrightarrow \max \\
4000x + 5000y &\leq 32000 \\
4x + 3y &\leq 24 \\
300x + 200y &\leq 2000 \\
200x + 100y &\leq 1600 \\
x &\geq 0 \\
y &\geq 0
\end{aligned}
$$

After plotting the set of feasible solutions, we can observe that the constraints related to the roofing iron and the concrete are not redundant, in the sense that they do not define part of the boundary of the feasible region. Otherwise, the coefficients of the objective function are the same as the coefficients of the first constraint, so when the objective function is drawn at the optimal level, it coincides with a boundary line of the feasible region. This means that all points on this border line represent optimal solutions. The only integer solution is the point (3;4) with \$32,000 as an optimum. But the builder has the same profit if he builds 2 of building A, and 4.8 of building B, which means that the fifth building is ready up to 80% at the end of the year. (When we accept only integer solutions we need other techniques for searching optimal solutions, see the topic of Integer Programming.)



2.1.4 No feasible solutions

A small clothing factory makes shirts and skirts. A profit of \$4 and \$3 is made from a shirt and a skirt respectively. A shirt requires 3 meters of material and a skirt 4, with only 12 meters available daily for a worker. It takes 5 hours of total time to make a shirt, and 2 hours to make a skirt. Supposing 10 working hours daily for a worker, the boss expects at least 4 garments per worker a day. How is it possible to maximize profit?

Solution: Let x be the number of the shirts and y be the number of the skirts. The problem can be formulated as follows:

$$\begin{aligned} 4x + 3y &\longrightarrow \max \\ 3x + 4y &\leq 12 \\ 5x + 2y &\leq 10 \\ x + y &\geq 4 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

When we try to draw the feasible region of the problem, we find, that there does not exist a point which will satisfy all constraints simultaneously. Hence the problem does not have a feasible solution, and so it can not have any optimal solution: the boss has unrealistic expectations for his workers.

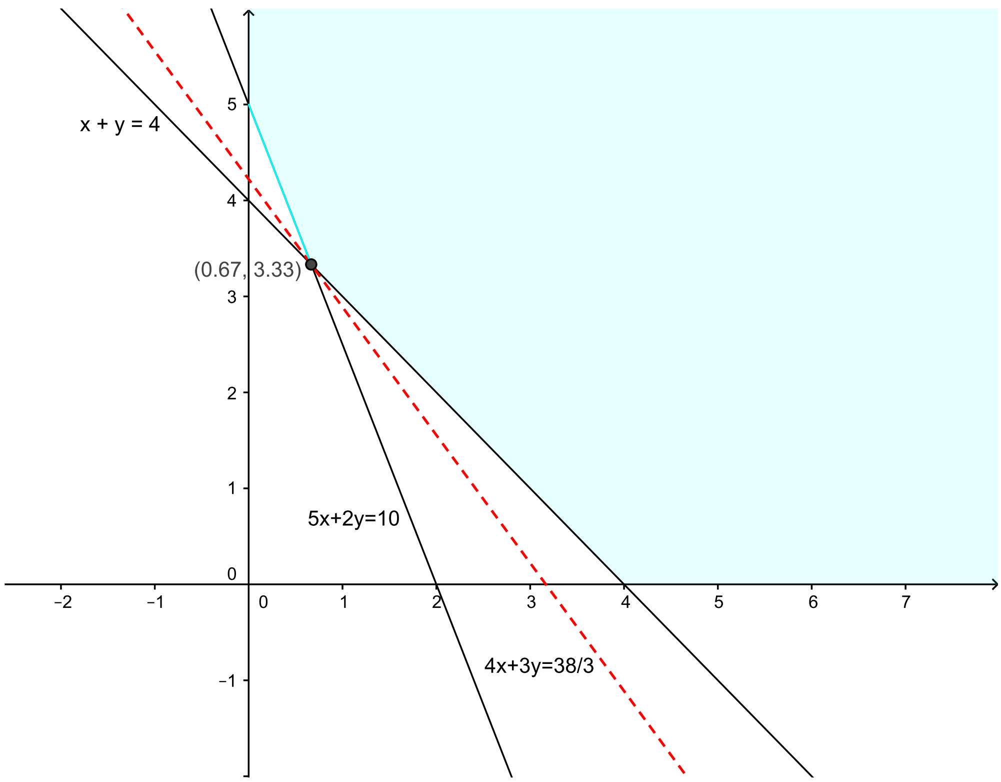
2.1.5 Unbounded feasible region

We reformulate the clothing factory problem studied before. Suppose that the boss of the factory got a lot of material at a bargain price, and lifted the limit concerning the usage of material.

However, he forces every employee to work at least 10 hours (and there is no upper bound for the working hours). The constraint of making at least 4 garments remains. In this new situation is it possible to achieve maximal profit?

$$
\begin{aligned}
4x + 3y &\longrightarrow \max \\
5x + 2y &\geq 10 \\
x + y &\geq 4 \\
x &\geq 0 \\
y &\geq 0
\end{aligned}
$$

The feasible region of the problem is not bounded. The line representing the objective function can be moved parallel to itself an arbitrary distance from the origin and still coincide with feasible points. Therefore the problem has no bounded maximal solution. (If the workers are able to work for infinitely long, the profit can be arbitrarily large). However, it is interesting to see, that the minimization problem subject to the constraints can be solved: the minimum profit is $\$\frac{38}{3}$ which can be achieved by making $\frac{2}{3}$ shirts and $\frac{10}{3}$ skirts.



2.2 The Standard Form of a Linear Programming Problem

In linear programming problems the objective function and all the constraints are linear functions of the decision variables $x_1, x_2, ..., x_n$. It is obvious that there are many variations of LP problems. Their objective can be maximization or minimization, the constraints can be given in equalities or inequalities form, and the variables can be restricted or unrestricted in sign. Fortunately it is not necessary to develop different methods for each class of problems. We will present a method which is suitable for solving the problems of one common and wide class: the problems which are given in standard form.

An LP problem is in *standard form* if it can be expressed as:

$$\begin{aligned} c^T x &\longrightarrow \max \\ Ax &= b, \quad (b \geq 0) \\ x &\geq 0 \end{aligned}$$

where

$$\begin{aligned} c &= (c_1, c_2, ..., c_n)^T && \text{the coefficients of the objective function} \\ x &= (x_1, x_2, ..., x_n)^T && \text{the vector of the decision variables} \\ A &= (a_{i,j})_{m \times n} && \text{the matrix containing the coefficients of the constraints} \\ b &= (b_1, b_2, ..., b_m)^T && \text{the right-hand-side constants} \\ 0 &= (0)_{n \times 1} && \text{a null vector with } n \text{ components.} \end{aligned}$$

The steps that transform an arbitrary LP problem into standard form are as follows:

1. A minimizing problem can be transformed into a maximizing problem by changing the signs of all coefficients of the objective function, because the function $f(x)$ has a minimum in the point x, when the function $-f(x)$ has its maximum.

2. Each negative right-hand-side constant can be made positive by multiplying the entire equality or inequality by -1.

3. An unrestricted decision variable x_i is replaced by $x_i' - x_i''$, where both new variables are nonnegative. If the constraint $x_j \leq 0$ is prescribed, we introduce a new nonnegative variable with $\widetilde{x}_j = -x_j$.

4. Each inequality can be made an equality by adding a nonnegative *slack variable* to a $\leq$ constraint, or subtracting a nonnegative *excess variable* from a $\geq$ constraint. More precisely, if a constraint is given in the form $a_{k1}x_1 + a_{k2}x_2 + ... + a_{kn}x_n \leq b_k$, we transform it into the equality

$$a_{k1}x_1 + a_{k2}x_2 + ... + a_{kn}x_n + u_k = b_k,$$

and if a constraint is given in the form $a_{l1}x_1 + a_{l2}x_2 + ... + a_{\ln}x_n \geq b_l$, instead of it we write

$$a_{l1}x_1 + a_{l2}x_2 + ... + a_{\ln}x_n - v_l = b_l$$

where the new constraints $u_k \geq 0$ and $v_l \geq 0$ are added to the problem.

Example 1. Determine the standard form of this general LP problem:

$$2x_1 + 3x_2 - 4x_3 \rightarrow \min$$

$$
\begin{aligned}
x_1 - x_2 &\le 5 \\
-x_1 + 2x_2 + x_3 &\le -2 \\
3x_1 - x_2 + x_3 &= 10 \\
x_1 \ge 0,\ x_2 \text{ unrestricted},\ & x_3 \le 0
\end{aligned}
$$

Solution:

1. Transform into a maximizing problem by multiplying the objective function by -1:

$$-2x_1 - 3x_2 + 4x_3 \to \max$$

2. The right-hand side of the second constraint is negative, so the entire row is multiplied by -1: $x_1 - 2x_2 - x_3 \ge 2$
3. We write $x_2' - x_2''$ instead of x_2, and $-\widetilde{x_3}$ instead of x_3
4. A new slack variable u_1 is added to the first constraint and an excess variable v_2 is subtracted from the second one.

The result is the following standard LP problem:

$$
\begin{aligned}
-2x_1 - 3x_2' + 3x_2'' - 4\widetilde{x_3} &\to \max \\
x_1 - x_2' + x_2'' + u_1 &= 5 \\
x_1 - 2x_2' + 2x_2'' + \widetilde{x_3} - v_2 &= 2 \\
3x_1 - x_2' + x_2'' - \widetilde{x_3} &= 10 \\
x_1 \ge 0,\ x_2' \ \ge 0, x_2'' \ \ge 0,\ \widetilde{x_3} \ge 0, u_1 \ge 0, v_2 \ge 0
\end{aligned}
$$

2.3 Geometrical Background of LP Problems

We studied the graphical solution of simple LP problems in the introductory chapters. We found that the feasible region of a two-variable problem is a polygon or an unbounded set with a finite number of corners. The objective function reaches its optimum in a corner of the feasible set or all the points of a boundary line provide optimal solutions of the problem. Similar geometrical considerations can be used in three-dimensional space, but a problem with 4 or more variables is not solvable in this way. In order to obtain a general solution method for the high-dimensional problems we introduce some basic notions and facts for extending the concepts applied in two-dimensional space.

Definition 3 (line segment)**.** Let x_1 and x_2 be two given points in the n-dimensional space $\mathbb{R}^{\ltimes}$. The line segment joining x_1 and x_2 contains all the points $x = \lambda x_1 + (1 - \lambda)\, x_2$, where $0 \le \lambda \le 1$.

Definition 4 (convex set)**.** A nonempty set C of $\mathbb{R}^{\ltimes}$ is called convex if the line segment joining any two points of C is also in C. That is, C is convex if $\lambda x_1 + (1 - \lambda)\, x_2 \in C$, for all $x_1, x_2 \in C$ and $0 \le \lambda \le 1$.

Definition 5 (bounded set)**.** A set $H \subset \mathbb{R}^{\ltimes}$ is called bounded if there exists a real number k, for which $||x_1 - x_2|| \le k$ for every $x_1, x_2 \in H$.

Definition 6 (extreme point)**.** Let $C \subset \mathbb{R}^{\ltimes}$ be a convex set. A point $E \in C$ is called an extreme point of C if E is not an inner point of some line segment in C, that is, E cannot be expressed as $E = \lambda x_1 + (1 - \lambda)\, x_2$, for some $0 < \lambda < 1$, $x_1 \neq x_2$, $x_1, x_2 \in C$.

Consider the feasible region $X = \{x : \ Ax = b, \ x \geq 0\}$ of an LP in the standard form. It can be proved, that the set X is convex. Assume, that in the system of linear equations $Ax = b$ we have n variables and k equations. In general $n \geq k$, because when transforming the LP into standard form, we introduce new variables, and the number of constraints is fixed. There are $\binom{n}{k}$ different ways of choosing a set of k *basic variables* and solve the system leaving the other variables to be 0. The solution, where we have k variables having nonzero values and $n - k$ variables with zero values is called a *basic solution* to the system. If all variables have nonnegative values in a basic solution, then the solution is called a *basic feasible solution.*

Theorem 1. *Any basic feasible solution of the system $Ax = b$ represents an extreme point of the feasible region $X = \{x : \ Ax = b, \ x \geq 0\}$.*

Definition 7. An LP is called

- feasible, if it has at least one feasible solution and infeasible, otherwise;
- optimal, if it has an optimal solution;
- unbounded, if it is feasible and its objective function is not bounded (from above for a maximization problem and from below for a minimization problem) in the feasible region.

Theorem 2.

- *If an LP is not optimal, it is either unbounded or infeasible.*
- *If the feasible region $X = \{x : \ Ax = b, \ x \geq 0\}$ of an LP is bounded, then the LP is optimal.*
- *If an LP is optimal, it either has a unique optimal solution or infinitely many optimal solutions.*
- *If an LP is optimal, then an optimal solution of the problem corresponds to an extreme point of X.*

Example 2. Consider the convex set given by the inequations

$$\begin{aligned} x_1 + 2x_2 &\leq 8 \\ 3x_1 + x_2 &\leq 9 \\ x_1, x_2 &\geq 0 \end{aligned} \tag{2.1}$$

Transform it in the form $Ax = b, \ x \geq 0$ and determine the basic feasible solutions of the system.

Solution: Two slack variables u_1 and u_2 are introduced, and we have the system:

$$\begin{aligned} x_1 + 2x_2 + u_1 &= 8 \\ 3x_1 + x_2 + u_2 &= 9 \\ x_1, x_2, u_1, u_2 &\geq 0 \end{aligned}$$

We have four variables. Two of them are chosen to be basic variables, the other two are nonbasic variables. The simplest way to determine the basic solutions is to delete two variables from the system (giving zero value to them) and solve the remaining system. For example, if $x_1 = 0$ and $x_2 = 0$, the value of the other two variables can be easily obtained as $u_1 = 8,\ u_2 = 9$.

Similarly, if $x_1 = 0$ and $u_1 = 0$, then $x_2 = 4$ and $u_2 = 5$.

If $x_1 = 0$ and $u_2 = 0$, then $x_2 = 9$ and $u_1 = -10$.

If $x_2 = 0$ and $u_1 = 0$, then $x_1 = 8$ and $u_2 = -15$.

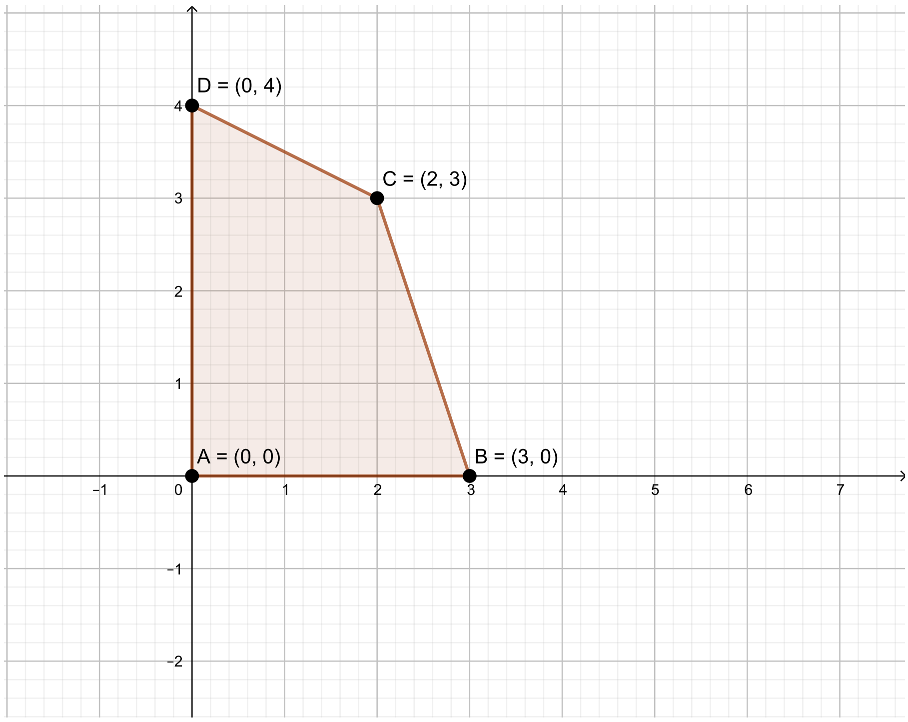
If $x_2 = 0$ and $u_2 = 0$, then $x_1 = 3$ and $u_1 = 5$,

and finally, if $u_1 = 0$ and $u_2 = 0$, then $x_1 = 2$ and $x_2 = 3$.

Make a table of our results:

x_1	0	0	0	8	3	2
x_2	0	4	9	0	0	3
u_1	8	0	-10	0	5	0
u_2	9	5	0	-15	0	0

We obtained $\binom{4}{2} = 6$ basic solutions, from which 4 are basic feasible solutions containing only nonnegative coordinates. The basic feasible solutions correspond to the extreme points of the convex set (2.1) as it is demonstrated in the following figure:



3 The Simplex Algorithm

In the previous chapter it was shown that if a standard LP problem has an optimal solution it corresponds to an extreme point of the feasible set X. The extreme points of X can be obtained by determining the basic feasible solutions of the system $Ax = b$. In order to determine the optimal solution of an LP problem we will use a very efficient technique, the simplex method. Its main steps are the following:

1. Select a basic feasible solution. This means geometrically that one of the corners (extreme points) of the feasible set X is chosen.

   Check whether the corner is optimal. If so, you have found the optimal solution and the algorithm terminates. Otherwise go to Step 2.

2. Choose a new basic feasible solution by replacing one of the basic variables by a nonbasic variables. The entering variable should be selected so as to improve the objective function value. Geometrically saying: move to an adjacent corner of X where a better objective function value is expected than it was previously.

This procedure is repeated until no improvement in objective function value can be made. When this happens the optimal solution has been found.

3.1 Simplex Method for Solving Normal LP Problems

For the sake of better understanding the simplex algorithm is first applied to a special LP problem which we call the *normal problem.* A normal problem is a maximizing problem with non-negative decision variables where every constraint is given by "$\leq$". When this problem is converted into standard form, we have to add a new u_i variable to the ith constraint. These new variables are called *slack variables*. The slack variables provide us with an initial basic solution of the problem, so we can start the simplex algorithm without any preparations.

When calculating manually the steps and operations of the algorithm are represented by the simplex tableau. This is actually the basis table of the system $Ax = b$, extended by a new row which contains the coefficients of the objective function at the beginning of the algorithm. This row will be called the *check row,* because its elements help in selecting the pivotal elements and identifying the end of the algorithm.

Let us consider the sandwich problem which is solved graphically in the previous chapter: A mother makes two types of sandwiches for the birthday party of her son. A salami sandwich consists of 3 g butter, 3 slices of egg and two slices of salami. A ham sandwich consists of 4 g butter, 2 slices of egg and a slice of ham. (Of course these ingredients are in between two slices of bread in both cases.) For making sandwiches 100 slices of salami, 40 slices of ham, 170 slices of egg and 220 g butter are available. There is plenty of bread, so no restriction is needed for bread. How many sandwiches of each type should be made to maximize the total number of sandwiches?

Denoting the variables by x_1 and x_2, the model of the problem is:

$$
\begin{aligned}
x_1 + x_2 &\longrightarrow \max \\
3x_1 + 4x_2 &\leq 220 \\
3x_1 + 2x_2 &\leq 170 \\
2x_1 &\leq 100 \\
x_2 &\leq 40 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

This is a normal problem, since every constraint is given by $\leq$. We need 4 slack variables to transform the problem into standard form:

$$
\begin{aligned}
x_1 + x_2 &\longrightarrow \max \\
3x_1 + 4x_2 + u_1 &\leq 220
\end{aligned}
$$

$$3x_1 + 2x_2 + u_2 \leq 170$$
$$2x_1 + u_3 \leq 100$$
$$x_2 + u_4 \leq 40$$
$$x_1, x_2, u_1, u_2 \geq 0$$

In the initial simplex tableau the slack variables form the basis:

	x_1	x_2	b
u_1	3	4	220
u_2	3	2	170
u_3	2	0	100
u_4	0	$\boxed{1}$	40
$-z$	1	1	0

The check row is labelled $-z$, because the last element of this row will show the opposite of the objective function value. Initially this value is 0, because all basic variables have zero objective value coefficients. In other words, at the beginning the variables x_1, x_2 are nonbasic variables, their values equal zero, so the objective function value $x_1 + x_2$ is also zero. Geometrically speaking: we are in the origin when we start examining the corners of the feasible set. The initial solution of the system $Ax = b$ can be read from the right-hand side, namely: $u_1 = 220, u_2 = 170, u_3 = 100, u_4 = 40$.

The next step is replacing one of the basic variables by a nonbasic variable so as to improve the objective function value. For this purpose we have to choose a suitable pivotal element. There are three important rules for selecting a pivotal element in the simplex tableau:

- The pivotal element must be positive. (A negative pivotal element would imply non-feasible solution, because the entire row is divided by the pivotal, resulting in a negative number in the right-hand side.)
- Choose the pivotal element from the column where the check row contains a positive number. (This choice guarantees the improvement of the objective function value.)
- After deciding the column of the pivotal element, we have to select the element which wins the ratio test. This means that for every element of the column we form the ratio of the right-hand-side constant to the element itself, and choose the element with the smallest ratio. (With this choice we avoid appearing negative elements in column b.)

If there is more than one option, try to choose the version which promises the easiest calculations.

Rules of pivoting (p denotes the pivotal element):

- write the reciprocal of the pivotal element instead of p
- the row of p is divided by p
- the column of p is divided by $-p$
- other elements are calculated the rectangle-rule: $x - \frac{ab}{p}$.

In our example, the check row allows the choice from both columns. If we decide on the first column, 2 must be the pivotal element, because the last is the smallest of the ratios $\frac{220}{3}, \frac{170}{3}, \frac{100}{2}$. In the second column 1 should be the winner of the ratio test: $\min\{\frac{220}{4}, \frac{170}{2}, \frac{40}{1}\} = \frac{40}{1}$. It is easier to calculate with 1, so we replace the variable u_4 by x_2. We use the rules for the calculation as previously discussed, extending the operations to the last row:

	x_1	u_4	b
u_1	$\boxed{3}$	-4	60
u_2	3	-2	90
u_3	2	0	100
x_2	0	1	40
$-z$	1	-1	-40

Note that if there is a 0 in the row of the pivotal element, the column of this 0 is not changed. Similarly, if there is a 0 in the column of the pivot element, the row of this 0 remains unchanged. (Explanation: when we apply the rectangle rule, 0 is to be subtracted from the number in question.)

The simplex tableau now shows a better solution then the initial one. We obtained the solution $x_1 = 0, x_2 = 40$, so 0 salami sandwiches and 40 ham sandwiches are made. (We are in the extreme point (0;40) of the polygon). However, the objective function value can be improved, because there is a positive number in the check row. We bring x_1 into the basis instead of u_1, because the first one is the smallest of the ratios $\frac{60}{3}, \frac{90}{3}$ and $\frac{100}{2}$:

	u_1	u_4	b
x_1	$\frac{1}{3}$	$\frac{-4}{3}$	20
u_2	-1	$\boxed{2}$	30
u_3	$-\frac{2}{3}$	$\frac{8}{3}$	60
x_2	0	1	40
$-z$	$-\frac{1}{3}$	$\frac{1}{3}$	-60

The present solution is $x_1 = 20, x_2 = 40$, the objective function value: $z = 60$. Since the check row contains again a positive number, the value z can be increased by replacing u_4 by u_2 :

	u_1	u_2	b
x_1	$-\frac{1}{3}$	$\frac{2}{3}$	40
u_4	$\frac{-1}{2}$	$\frac{1}{2}$	15
u_3	$-\frac{2}{3}$	$-\frac{4}{3}$	60
x_2	$\frac{1}{2}$	$-\frac{1}{2}$	25
$-z$	$-\frac{1}{6}$	$-\frac{1}{6}$	-65

This tableau is optimal, because there are no more positive elements in the check row. The algorithm terminates with the following result: the optimum is 65, which is achieved in the point (40;25), as seen previously by the graphical solution of the problem.

3.2 A Three-Variable Example of the Simplex Algorithm

The next problem contains three decision variables, so it cannot be represented in two-dimensional space.

A plant manufactures three types of vehicle: automobiles, trucks and vans, on which the company makes a profit of \$4000, \$6000 and \$3000, respectively, per vehicle. The plant has three main departments: parts, assembly and finishing. The next table shows how many hours a vehicle spends in the different departments, and what the weekly capacities in these departments are:

	automobile	truck	van	capacity
parts	50	40	30	240
assembly	40	30	20	200
finishing	20	40	10	160

How many of each type of vehicle should the company manufacture in order to maximize profit for a weekly period?

Solution: Let x_1 = the number of automobiles manufactured, x_2 = the number of trucks manufactured and x_3 = the number of vans manufactured.

After dividing the profits by 1,000, and the constraints by 10 we have the following problem:

$$\begin{gathered} 4x_1 + 6x_2 + 3x_3 \longrightarrow \max \\ 5x_1 + 4x_2 + 3x_3 \leq 24 \\ 4x_1 + 3x_2 + 2x_3 \leq 20 \\ 2x_1 + 4x_2 + x_3 \leq 16 \\ x_1, x_2, x_3 \geq 0 \end{gathered}$$

We introduce three slack variables and the problem becomes:

$$\begin{gathered} 4x_1 + 6x_2 + 3x_3 \longrightarrow \max \\ 5x_1 + 4x_2 + 3x_3 + u_1 = 24 \\ 4x_1 + 3x_2 + 2x_3 + u_2 = 20 \\ 2x_1 + 4x_2 + x_3 + u_3 = 16 \\ x_1, x_2, x_3, u_1, u_2, u_3 \geq 0 \end{gathered}$$

The initial simplex tableau shows the first solution: the plant manufactures 0 vehicles with 0 profit:

	x_1	x_2	x_3	b
u_1	5	4	3	24
u_2	4	3	2	20
u_3	2	$\boxed{4}$	1	16
$-z$	4	6	3	0

We choose the pivotal element from the second column, because variable x_2 promises the greatest profit:

	x_1	u_3	x_3	b
u_1	3	-1	$\boxed{2}$	8
u_2	$\frac{5}{2}$	$-\frac{3}{4}$	$\frac{5}{4}$	8
x_2	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	4
$-z$	1	$-\frac{3}{2}$	$\frac{3}{2}$	-24

In the next step we replace u_1 by x_3:

	x_1	u_3	u_1	b
x_3	$\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	4
u_2	$\frac{5}{8}$	$-\frac{1}{8}$	$-\frac{5}{8}$	3
x_2	$\frac{1}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	3
$-z$	$-\frac{5}{4}$	$-\frac{3}{4}$	$-\frac{3}{4}$	-30

The algorithm terminates, because there is no positive element in the check row. The optimal solution is: $x_1 = 0, x_2 = 3, x_3 = 4$. The optimum is 30 in the tableau, that is the maximum profit is $30,000, which can be achieved by manufacturing 3 trucks and 4 vans in a week. The value of the slack variable u_2 equals 3 in the optimal tableau, meaning that there are 30 hours of unused capacity in assembly. The other two departments work at full capacity because the slack variables u_1 and u_2 are zeros.

3.3 Further Examples

The next problem has multiple optimal solutions. Now we examine how this case can be identified when applying the simplex algorithm.

$$2x_1 + 2x_2 \longrightarrow \max$$
$$x_1 + x_2 \leq 70$$
$$x_1 \leq 50$$
$$x_2 \leq 40$$
$$x_1 \geq 0, x_2 \geq 0$$

The steps of the algorithm:

	x_1	x_2	b
u_1	1	1	70
u_2	1	0	50
u_3	0	[1]	40
$-z$	2	2	0

	x_1	u_3	b
u_1	[1]	-1	30
u_2	1	0	50
x_2	0	1	40
$-z$	2	-2	-80

	u_1	u_3	b
x_1	1	-1	30
u_2	-1	1	20
x_2	0	1	40
$-z$	-2	0	-140

There is no more positive number in the check row, the solution $x_1 = 30, x_2 = 40$ is optimal and the optimum is 140. However the variable u_3 has a zero check row coefficient, indicating that the objective function value would remain unchanged if u_3 was brought into the basis. In order to check it we replace u_2 by u_3:

	u_1	u_2	b
x_1	0	1	50
u_3	-1	1	20
x_2	1	-1	20
$-z$	-2	0	-140

We have a new optimal solution $x_1 = 50, x_2 = 20$ with the same optimum as before. Actually there are infinitely many optimal solutions: any point on the line segment joining the two basic optimal solutions is optimal. Formally, the optimal solutions are the points $\lambda(30; 40) + (1 - \lambda)(50; 20)$ where $\lambda \in [0; 1]$.

The following problem does not have an optimal solution, because the objective function is not bounded on the feasible set.

$$3x_1 + 2x_2 \longrightarrow \max$$
$$\tfrac{1}{2}x_1 - x_2 \leq 1$$
$$2x_1 - x_2 \leq 7$$
$$x_1 \geq 0, x_2 \geq 0$$

This problem was also studied before, we now attempt to solve it by the simplex method:

	x_1	x_2	b
u_1	[$\frac{1}{2}$]	-1	1
u_2	2	-1	7
$-z$	3	2	0

	u_1	x_2	b
x_1	2	-2	2
u_2	-4	[3]	3
$-z$	-6	8	-6

	u_1	u_2	b
x_1	$-\frac{2}{3}$	$\frac{2}{3}$	4
x_2	$-\frac{4}{3}$	$\frac{1}{3}$	1
$-z$	$\frac{14}{3}$	$-\frac{8}{3}$	-14

From the last table it can be seen that u_1 should enter the basis next. The problem is that all of the coefficients are negative in u_1's column, so it is not possible to choose a pivotal element. A positive check row coefficient whose column entries are all non-positive indicates that the objective function can have arbitrarily large value and there is no bounded optimal solution of the problem.

3.4 The Two-Phase Simplex Method

Until now we used the simplex algorithm for solving normal problems where all constraints were of the $\leq$ type. In this case the introduced slack variables formed an initial basis, and the simplex method can be started conveniently. With constraints $=$ or $\geq$ the procedure is different, because an initial basic solution is not known as it was before. The first phase of the method aims to create an initial feasible basis so that the simplex algorithm can be used. After converting the problem into standard form, a new artificial variable is added to the left-hand side of each constraint equation which was of the $=$ or $\geq$ type. These new variables are denoted by u_i^* and they are also constrained to be non-negative. Then a new objective function is introduced as a sum of the artificial variables. This creates a new problem in which the secondary objective function is to be minimized. If the optimal solution value of the new problem is greater than zero, the original problem does not have a feasible solution. Otherwise we obtain a basic and feasible solution of the original problem because all the artificial variables have value zero. The obtained solution can be used as a starting solution for further iterations of the simplex method. The second phase consists of these iterations.

For example consider the following general LP problem:

$$\begin{aligned} 2x_1 + 6x_2 + 2x_3 &\longrightarrow \max \\ x_2 + x_3 &\leq 160 \\ -x_1 + x_2 + x_3 &= 100 \\ x_1 + x_2 + x_3 &\geq 180 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

In order to convert it into standard form, a slack variable u_1 is added to the left-hand side of the first constraint and an excess variable v_3 is subtracted from the third one. In addition two artificial variables are introduced, and they are added to the left-hand side of the last two constraints. The new problem can be formulated as:

$$\begin{aligned} 2x_1 + 6x_2 + 2x_3 &\longrightarrow \max \\ x_2 + x_3 + u_1 &= 160 \\ -x_1 + x_2 + x_3 + u_2^* &= 100 \\ x_1 + x_2 + x_3 - v_3 + u_3^* &= 180 \\ x_1, x_2, x_3, u_1, u_2^*, u_3^*, v_3 &\geq 0 \end{aligned}$$

The initial simplex tableau contains an extra row, labeled z^*. The elements of this row are obtained by adding the rows of the artificial variables. First we have to find the optimum of the secondary objective function.

	x_1	x_2	x_3	v_3	
u_1	0	1	1	0	160
u_2^*	-1	$\boxed{1}$	1	0	100
u_3^*	1	1	1	-1	180
$-z$	2	6	2	0	0
z^*	0	2	2	-1	280

We do not need artificial variables in the basis, so variable x_2 is entered and u_2^* is removed:

	x_1	u_2^*	x_3	v_3	
u_1	1	-1	0	0	60
x_2	-1	1	1	0	100
u_3^*	$\boxed{2}$	-1	0	-1	80
$-z$	8	-6	-4	0	-600
z^*	2	-2	0	-1	80

Now replace u_3^* by x_1:

	u_3^*	u_2^*	x_3	v_3	
u_1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	20
x_2	$\frac{1}{2}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	140
x_1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	40
$-z$	-4	-2	-4	4	-920
z^*	-1	-1	0	0	0

The first phase is finished because the artificial variables are removed from the basis (their value is 0) and z^* is also 0. The simplex tableau contains a feasible solution of the original problem as follows: $x_1 = 40, x_2 = 140, x_3 = 0$, and the objective function value is 920. We can leave the last row and the columns belonging to u_3^* és u_2^* from the tableau:

	x_3	v_3	
u_1	0	$\boxed{\frac{1}{2}}$	20
x_2	1	$-\frac{1}{2}$	140
x_1	0	$-\frac{1}{2}$	40
$-z$	-4	4	-920

The check row contains a positive number, indicating that a program can be improved by bringing v_3 into the basis:

	x_3	u_1	
v_3	0	2	40
x_2	1	1	160
x_1	0	1	60
$-z$	-4	-8	-1080

Every check row coefficient is negative, the second phase is finished. The optimal solution of the problem: $x_1 = 60, x_2 = 160, x_3 = 0$ and the optimum is 1080.

4 Duality and Sensitivity Analysis

4.1 Dual of a Normal Problem

In the previous chapters we studied a lot of LP problems, and it became obvious that the difficulty of their solution mainly depends on the number and the type of their constraints. It would be useful if we could reduce the number of constraints in relatively large problems. This is often possible by constructing a new LP problem, the so-called *dual problem*. When we talk about the dual problem, the original problem is referred as the *primal*. We should decide which problem can be solved more easily, because the final simplex tableau contains the solution of both problems.

First we construct the dual pair of a normal problem and at the same time give an interpretation of the dual problem. Recall that a normal problem often models the following situation:

A company produces n types of products using m types of sources which are available in a restricted amount. The technological matrix is given, its element a_{ij} is the amount of the source i which is needed for producing a unit of product j. The available amount (capacity) of source i is b_i, the profit from a unit of product j is c_j. Determine the amount of each type of products so as to maximize the company's profit.

Denoting by x_i the amount of product i to be produced, we obtain the following problem:

$$\begin{aligned} c_1x_1 + c_2x_2 + ... + c_nx_n &\longrightarrow \max \\ a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &\leq b_1 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n &\leq b_m \\ x_{1,}x_2, ..., x_n &\geq 0 \end{aligned}$$

Using vector-matrix denotations the model can be set more briefly:

$$\begin{aligned} c^Tx &\longrightarrow \max \\ Ax &\leq b \\ x &\geq 0 \end{aligned}$$

Suppose that a corporation is considering hiring the capacities of the company. Let y_i be the hiring rate of a unit of source i, $\quad i = (1, 2, ..., m)$. Of course all hiring costs have to be nonnegative. It is not worth hiring the sources if the revenue is less than the present profit, so the following inequalities must be valid:

$$\begin{aligned} a_{11}y_1 + a_{21}y_2 + ... + a_{m1}y_m &\geq c_1 \\ &\vdots \\ a_{1n}y_1 + a_{2n}y_2 + ... + a_{mn}y_m &\geq c_n \end{aligned}$$

However, the corporation wishes to minimize the total hiring cost it has to pay, so it wants to keep the amount $b_1y_1 + b_2y_2 + ...b_my_m$ as low as possible.

The problem can be summarized as follows:

$$\begin{aligned} b^Ty &\longrightarrow \min \\ A^Ty &\geq c \end{aligned}$$

$$y \geq 0$$

and this problem is called the *dual problem* of the original normal problem.

Theorem 3 (Strong Duality Theorem)**.** *If either the primal or the dual problem has an optimal solution, then so does the other, and the optimums coincide.*

Example 3. Determine the dual of the following normal problem. Solve the primal problem by the simplex method. Using the final simplex tableau give the optimal solution to the dual problem.

$$\begin{aligned} 7x_1 + 9x_2 + 5x_3 &\longrightarrow \max \\ x_1 + x_2 + 2x_3 &\leq 5 \\ 2x_1 + x_2 + 3x_3 &\leq 11 \\ 2x_1 + 3x_2 + 2x_3 &\leq 12 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Solution: There are 3 constraints in the primal problem which indicate 3 variables in the dual. The coefficients in constraint i of the dual are given by the coefficients of the variable x_i in the primal constraints. The dual right-hand-side constants are equal to the primal objective function coefficients, and vice versa. The dual is a minimizing problem and all constraints are of $\geq$ type. Dual variables are also nonnegative. So the dual problem is expressed as

$$\begin{aligned} 5y_1 + 11y_2 + 12y_3 &\longrightarrow \min \\ y_1 + 2y_2 + 2y_3 &\geq 7 \\ y_1 + y_2 + 3y_3 &\geq 9 \\ 2y_1 + 3y_2 + 2y_3 &\geq 5 \\ y_1, y_2, y_3 &\geq 0 \end{aligned}$$

We solve the primal problem by the simplex method:

	x_1	x_2	x_3	b
u_1	$\boxed{1}$	1	2	5
u_2	2	1	3	11
u_3	2	3	2	12
$-z$	7	9	5	0

	u_1	x_2	x_3	b
x_1	1	1	2	5
u_2	-2	-1	-1	1
u_3	-2	$\boxed{1}$	-2	2
$-z$	-7	2	-9	-35

	u_1	u_3	x_3	b
x_1	3	-1	4	3
u_2	-4	1	-3	3
x_2	-2	1	-2	2
$-z$	-3	-2	-5	-39

The final tableau is optimal, the primal optimal solution is $x_1 = 3, x_2 = 2, x_3 = 0, z_{\max} = 39$.

In this case the dual problem can be solved by the two-phase method, which would need more effort. However, this procedure can be avoided, because the dual optimal solution can be read from the final simplex tableau of the primal. The optimums of the primal and dual are equal, so the minimum of the dual objective function is 39. The dual optimal solution can be determined by the slack variables of the primal: y_i is the opposite of the check row coefficient belonging to u_i. If a variable u_i has remained in the basis, the corresponding y_i has the value of zero in the dual optimal solution. So the optimal solution of the dual problem is $y_1 = 3, y_2 = 0, y_3 = 2$, and the optimum is 39, as mentioned before.

4.2 Dual of a General LP Problem

In general LP problems the constraints can be given by equalities or inequalities and the signs of the variables can be restricted in various ways. Using the following table we can construct the corresponding dual problem to an arbitrary LP problem.

	maximizing problem	minimizing problem	
variables	≥ 0 unrestricted ≤ 0	$\geq$ $=$ $\leq$	constraints
constraints	$\leq$ $=$ $\geq$	≥ 0 unrestricted ≤ 0	variables

Some hints to the usage of the table:

The dual of a maximizing problem is a minimizing problem and vice versa.

Each primal constraint corresponds to a dual variable and each primal variable corresponds to a dual constraint. The type of the constraints and the restrictions for the variables are strongly connected. For instance if a constraint in a maximizing problem is given by $\leq$, then the corresponding dual variable must be nonnegative. If a variable in a maximizing problem is unrestricted, then the corresponding constraint is an equality, and so on.

Example 4. Construct the dual of the following problem:

$$\begin{aligned} 2x_1 - x_2 + 5x_3 - 4x_4 &\longrightarrow \max \\ x_1 + x_2 - x_3 - 2x_4 &= 60 \\ x_1 + x_2 - 2x_3 + 3x_4 &\geq 100 \\ x_1 + x_3 + 2x_4 &\leq 80 \\ x_1 \geq 0, x_2 \leq 0, x_3 &\text{ unrestricted}, x_4 \geq 0 \end{aligned}$$

Solution: The dual problem is a minimizing problem with 3 variables and 4 constraints, because the primal has 3 constraints and 4 variables.

- $x_1 \geq 0$ and $x_4 \geq 0$ is prescribed in the primal, so the first and fourth dual constraints are of $\geq$ type,
- $x_2 \leq 0$ in the primal, the second dual constraint is of $\leq$ type,
- and the primal variable x_3 is unrestricted, implying an equality in the third dual constraint,
- the first primal constraint is an equality, which is why the dual variable y_1 is unrestricted,
- the second primal constraint is given by $\geq$ implying $y_2 \leq 0$,
- because of the $\leq$ in the third primal constraint the dual variable y_3 must be nonnegative.

The obtained dual problem is:

$$\begin{aligned} 60y_1 + 100y_2 + 80y_3 &\longrightarrow \min \\ y_1 + y_2 + y_3 &\geq 2 \\ y_1 + y_2 &\leq -1 \\ -y_1 - 2y_2 + y_3 &= 5 \\ -2y_1 + 3y_2 + 2y_3 &\geq -4 \\ y_1 \text{ unrestricted}, y_2 &\leq 0, y_3 \geq 0 \end{aligned}$$

If a primal problem is solved by the two-phase simplex method, we usually remove the u^* columns before starting the second phase. However, when a dual optimal solution should be determined, it is better to save these columns. For example consider the following general problem:

$$
\begin{aligned}
5x_1 - 2x_2 + 8x_3 &\longrightarrow \max \\
x_1 + 2x_2 + x_3 &\leq 50 \\
x_1 + x_2 &= 40 \\
x_1 + x_2 + 2x_3 &\geq 20 \\
x_1, x_2, x_3 &\geq 0
\end{aligned}
$$

Only the starting and the final tableau are presented, the others are left to the reader for the sake of practice:

	x_1	x_2	x_3	v_3	b
u_1	1	2	1	0	50
u_2^*	1	1	0	0	40
u_3^*	1	1	2	-1	20
$-z$	5	-2	8	0	0
z^*	2	2	2	-1	60

	x_2	u_1	b	u_2^*	u_3^*
x_3	1	1	10	-1	0
v_3	2	2	40	-1	-1
x_1	1	0	40	1	0
-z	-15	-8	-280	3	0

The columns of the variables u^* are copied to the right-hand side of the tableau. The optimal solution to the primal: $x_1 = 40, x_2 = 0, x_3 = 10, z_{\max} = 280$.

The dual problem is formulated as

$$
\begin{aligned}
50y_1 + 40y_2 + 20y_3 &\longrightarrow \min \\
y_1 + y_2 + y_3 &\geq 5 \\
2y_1 + y_2 + y_3 &\geq -2 \\
y_1 + 2y_3 &\geq 8 \\
y_1 \geq 0, y_2, \text{unrestricted}, y_3 &\leq 0
\end{aligned}
$$

The solution to the dual can be read from the optimal tableau of the primal. The variable y_2 was unrestricted, in this case its sign in the optimal solution is determined, so as the dual objective function has the optimum equalling the optimum of the primal problem. This remark explains the dual optimal solution, which is $y_1 = 8, y_2 = -3, y_3 = 0$ and $z_{\min} = 280$.

4.3 Sensitivity Analysis for the right-hand-side Constants

Duality can be used to answer questions related to the sensitivity of the solution. We would often like to know what happens with the solution if the parameters of the problem are changed. Obviously these questions can be answered by solving the problem from the beginning, but it is not necessary, at least in case of minor changes. We will make the sensitivity analysis based on the dual variables without recalculating all simplex tableaus.

Sensitivity analysis considers the range within which some coefficients of the problem can vary without affecting the optimal solution. It can begin after an LP problem has been solved, and deals with the impact of changing one parameter at a time. First we study the case when one of the right-hand- side constants has been modified.

For the sake of better understanding consider the following example:

Example 5. A company produces 3 types of products (T_1, T_2, T_3) by using 3 types of employees (E_1, E_2, E_3). The next table shows the needs of each employee for producing a unit of each product, the available amounts of employees and the unit prices of the products.

	T_1	T_2	T_3	capacity
E_1	1	1	2	5
E_2	2	1	3	11
E_3	2	3	2	12
price	7	9	5	

The problem is to find the combination of products so as to maximize profit, as usual.

After formulating the problem, it may be familiar, because it was solved as an introductory example in a previous chapter:

$$\begin{aligned} 7x_1 + 9x_2 + 5x_3 &\longrightarrow \max \\ x_1 + x_2 + 2x_3 &\le 5 \\ 2x_1 + x_2 + 3x_3 &\le 11 \\ 2x_1 + 3x_2 + 2x_3 &\le 12 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

The optimal solution is: $x_1 = 3, x_2 = 2, x_3 = 0, z_{\max} = 39$.

Suppose that the right-hand side value of the first constraint is altered from 5 to $5 + \lambda$. For what value of λ will the present solution remain feasible and optimal? (We do not want changes in the structure of the solution: x_1 and x_2 should remain the basic variables.) In order to answer the question we start the simplex algorithm with $5 + \lambda$ instead of 5:

	x_1	x_2	x_3	b
u_1	$\boxed{1}$	1	2	$5+\lambda$
u_2	2	1	3	11
u_3	2	3	2	12
$-z$	7	9	5	0

	u_1	x_2	x_3	b
x_1	1	1	2	$5+\lambda$
u_2	-2	-1	-1	$1-2\lambda$
u_3	-2	$\boxed{1}$	-2	$2-2\lambda$
$-z$	-7	2	-9	$-35-7\lambda$

	u_1	u_3	x_3	b
x_1	3	-1	4	$3+3\lambda$
u_2	-4	1	-3	$3-4\lambda$
x_2	-2	1	-2	$2-2\lambda$
$-z$	-3	-2	-5	$-39-3\lambda$

Comparing the final tableau with the one obtained in the previous chapter, we can observe that only the last column has been modified. For the optimality of this tableau the right-hand side coefficients should be nonnegative, that is, the following inequalities are to be satisfied at the same time:

$$\begin{array}{l} 3+3\lambda \ge 0 \\ 3-4\lambda \ge 0 \\ 2-2\lambda \ge 0 \end{array} \text{, which means } \begin{array}{l} \lambda \ge -1 \\ \lambda \le \frac{3}{4} \\ \lambda \le 1 \end{array} \text{, that is, } -1 \le \lambda \le \tfrac{3}{4}.$$

We can read the new solution from the final simplex tableau: if λ is between -1 and 0.75, then the optimal solution of the modified problem is $x_1 = 3 + 3\lambda, x_2 = 2 - 2\lambda, x_3 = 0, z_{\max} = 39 + 3\lambda$. For example if the available amount of the first labor is increased by 0.5, then we obtain the following optimal solution: $x_1 = 4.5, x_2 = 1, z_{\max} = 40.5$.

As it was mentioned earlier, it is not necessary to do the calculations from the beginning. Suppose that the right-hand-side constant of constraint i is changed from b_i to $b_i + \lambda$. Solve the system of inequalities $\overline{b_i} + \lambda \overline{u_i} \geq 0$, where $\overline{b_i}$ is the right-hand side element and $\overline{u_i}$ is the u_i-coefficient of the final simplex tableau. The solution gives limits for parameter λ within which the structure of the original solution remains unchanged. The value x_i in the new solution can be determined by $x_i = \overline{b_i} + \lambda \overline{u_i}$ if x_i is a basic variable, otherwise x_i remains zero.

For example, suppose that the right-hand side value of the third constraint changes to $12+\lambda$ from 12. We have three inequalities to solve:

$$\begin{array}{lll} 3-\lambda \geq 0 & \lambda \leq 3 & \\ 3+\lambda \geq 0 \;\rightarrow & \lambda \geq -3 & \rightarrow -2 \leq \lambda \leq 3, \\ 2+\lambda \geq 0 & \lambda \geq -2 & \end{array}$$

and then the optimal solution of the problem is $x_1 = 3-\lambda, x_2 = 2+\lambda, x_3 = 0, z_{\max} = 39+2\lambda$.

Observe the different range of changes in the objective function value when modifying the first and the third capacities. The change depends on the check row coefficient corresponding to the variable u_i which is actually the optimal solution of the dual problem. In economics this important parameter is referred as the *shadow price.* The shadow price of a constraint is the amount that the objective function value would change if the named constraint changed by one unit. The shadow price is valid up to the allowable increase or decrease in the constraint.

We have not dealt with the case of the second constraint yet. Suppose that the right-hand side constant of the second constraint is changed from 11 to $11 + \lambda$. The variable u_2 is in the basis, its coefficients can be represented by the unit vector $(0, 1, 0)^T$ which is not presented in the simplex tableau. So we have only one inequality for λ, namely $3 + \lambda \geq 0$, yielding $\lambda \geq -3$. This means that the capacity of the second source can be decreased at most 3 units (and can be increased by an arbitrary amount). The shadow price is 0, any change in capacity leaves the optimum unchanged, the solution remains: $x_1 = 3, x_2 = 2, x_3 = 0, z_{\max} = 39$.

4.4 Sensitivity Analysis for the Objective Function Coefficients

Suppose that one of the objective function coefficients is changed (for example the price of a product is decreased). What is the range of this change for which the optimal solution of the original problem will remain optimal? Let us consider again the previous problem:

$$\begin{aligned} 7x_1 + 9x_2 + 5x_3 &\longrightarrow \max \\ x_1 + x_2 + 2x_3 &\leq 5 \\ 2x_1 + x_2 + 3x_3 &\leq 11 \\ 2x_1 + 3x_2 + 2x_3 &\leq 12 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Suppose that the first coefficient in the objective function changes from 7 to $7 + \lambda$. Follow the impact of this modification by recalculating the simplex tableaus with $7 + \lambda$ instead of 7:

	x_1	x_2	x_3	b
u_1	$\boxed{1}$	1	2	5
u_2	2	1	3	11
u_3	2	3	2	12
$-z$	$7+\lambda$	9	5	0

	u_1	x_2	x_3	b
x_1	1	1	2	5
u_2	-2	-1	-1	1
u_3	-2	$\boxed{1}$	-2	2
$-z$	$-7-\lambda$	$2-\lambda$	$-9-2\lambda$	$-35-5\lambda$

	u_1	u_3	x_3	b
x_1	3	-1	4	3
u_2	-4	1	-3	3
x_2	-2	1	-2	2
$-z$	$-3-3\lambda$	$-2+\lambda$	$-5-4\lambda$	$-39-3\lambda$

Only the check row coefficients have been changed (this is not surprising, because the feasible region is unaltered). The original optimal solution remains optimal if the check row has no positive elements. The inequalities $-3-3\lambda \leq 0, -2+\lambda \leq 0$ and $-5-4\lambda \leq 0$ imply $-1 \leq \lambda \leq 2$. This means that the unit price of product T_1 can be moved between 6 and 9 without changing the optimal solution $x_1 = 3, x_2 = 2, x_3 = 0$. Obviously the optimal value of the objective function should be modified: it changes from 39 to $39 + 3\lambda$.

In general, if coefficient i of the objective function is changed from c_i to $c_i + \lambda$ then solve the system of inequalities $\overline{z_i} - \lambda \overline{x_i} \leq 0$, where $\overline{z_i}$ is the check row coefficient i and $\overline{x_i}$ is the x_i-coefficient of the final simplex tableau.

For instance, if the second coefficient of the objective function is changed from 9 to $9 + \lambda$, we have the inequalities $-3+2\lambda \leq 0$, $-2-\lambda \leq 0$ and $-5+2\lambda \leq 0$. The solution is $-2 \leq \lambda \leq \frac{3}{2}$. The optimal solution of the problem remains $x_1 = 3, x_2 = 2, x_3 = 0$ if the price of T_2 is changed by λ, where $-2 \leq \lambda \leq \frac{3}{2}$. The new optimum will be $39 + 2\lambda$.

If the third coefficient of the objective function is changed, the calculations are even simpler, because the coefficients belonging to x_3 form a unit vector (x_3 is a nonbasic variable). Only the condition $-5 - \lambda \leq 0$ should be satisfied, that is, $\lambda \geq -5$. This means that the price of T_3 can be an arbitrary nonnegative value. To explain this result, we need to take into consideration that the structure of the solution must remain unchanged, i.e. product T_3 is not involved in production, so its price has no impact on the optimum.

5 Linear-Fractional Programming

Linear-fractional programming (LFP) can be considered as a generalization of linear programming. The objective function of a linear-fractional programming problem is not linear but a ratio of two linear functions. A linear program can be regarded as a special case of a linear-fractional program in which the denominator is the constant function one. An LFP problem can be transformed into an LP problem, which then can be solved by the simplex method.

Consider a factory which produces x_i units of its product i. The cost of producing a unit of product i is denoted by d_i, and the price for which it can be sold is c_i. The problem is to find the highest ratio of income to cost showing the efficiency of the production. In order to formulate the problem let $x = (x_1, x_2, ..., x_n)^T$, $c = (c_1, c_2, ..., c_n)^T$ and $d = (d_1, d_2, ..., d_n)^T$. The standard form of the linear-fractional programming problem is defined by

$$\frac{c^T x + c_0}{d^T x + d_0} \longrightarrow \max$$
$$Ax = b$$
$$x \geq 0$$

We will assume that the set of the feasible solutions is non-empty and bounded. In addition the denominator of the objective function is assumed to be positive in the feasible region (the coefficients of the denominator can be multiplied by -1, if needed). These assumptions guarantee the existence of an optimal solution of the LFP problem.

In order to convert the LFP problem into an LP problem the nominator and the denominator of the objective function and all constraints should be multiplied by a positive unknown t. After this transformation we have the following problem:

$$\frac{c^T xt + c_0 t}{d^T xt + d_0 t} \longrightarrow \max$$
$$Axt - bt = 0$$

$$x \geq 0, t \geq 0$$

Next, new variables are introduced by the definition $y = xt$, and the denominator of the objective function is required to be 1. With this restriction it is enough to optimize the nominator of the objective function. The obtained problem is a linear programming problem as follows:

$$\begin{aligned} c^T c + c_0 t &\longrightarrow \max \\ Ay - bt &= 0 \\ d^T y + d_0 t &= 1 \\ y \geq 0, t &\geq 0 \end{aligned}$$

Example 6. Solve the linear-fractional programming problem given by

$$\begin{aligned} \frac{2x_1 + x_2 + x_3 - 1}{x_1 + x_2 + 2x_3 + 1} &\to \max \\ 2x_1 + 2x_2 &\leq 6 \\ x_1 + x_2 + x_3 &\leq 4 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Solution: The nominator and the denominator of the objective function and both constraints are multiplied by t. New variables are defined by $y_i = x_i t$ $(i = 1, 2, 3)$. The denominator of the objective function is restricted to be 1. Then the problem is:

$$\begin{aligned} 2y_1 + y_2 + y_3 - t &\to \max \\ 2y_1 + 2y_2 - 6t &\leq 0 \\ y_1 + y_2 + y_3 - 4t &\leq 0 \\ y_1 + y_2 + 2y_3 + t &= 1 \\ y_1, y_2, y_3, t &\geq 0 \end{aligned}$$

This LP problem can be solved by the two-phase simplex method, because there is an equality between the constraints:

The tableaus of the first phase:

	y_1	y_2	y_3	t	
u_1	2	2	0	-6	0
u_2	1	1	1	-4	0
u_3^*	1	1	2	$\boxed{1}$	1
$-z$	2	1	1	-1	0
z^*	1	1	2	1	1

	y_1	y_2	y_3	u_3^*	
u_1	8	8	12	6	6
u_2	5	5	9	4	4
t	1	1	2	1	1
$-z$	3	2	3	1	1
z^*	0	0	0	-1	0

As the secondary objective function became zero, the first phase is finished; we found an initial solution. The last row and the column of u_3^* are removed, and we can start the second phase:

	y_1	y_2	y_3	
u_1	$\boxed{8}$	8	12	6
u_2	5	5	9	4
t	1	1	2	1
$-z$	3	2	3	1

	u_1	y_2	y_3	
y_1	$\frac{1}{8}$	1	$\frac{3}{2}$	$\frac{3}{4}$
u_2	$-\frac{5}{8}$	0	$\frac{3}{2}$	$\frac{1}{4}$
t	$-\frac{1}{8}$	0	$\frac{1}{2}$	$\frac{1}{4}$
$-z$	$-\frac{3}{8}$	-1	$-\frac{3}{2}$	$-\frac{5}{4}$

All check row elements are negative; an optimal solution is found for the transformed problem: $y_1 = \frac{3}{4}; y_2 = 0; y_3 = 0; t = \frac{1}{4}$ and $z_{\max} = \frac{5}{4}$.

The optimal solution of the original LFP problem: $x_1 = \frac{y_1}{t} = 3; x_2 = \frac{y_2}{t} = 0; x_3 = \frac{y_3}{t} = 0$. The optimal value of the objective function is $\frac{5}{4}$, equal to the optimum of the LP problem.

6 The Transportation Problem

The transportation problem is a special linear programming problem. Because of its structure there are more efficient methods for solving it than the simplex algorithm. Among these techniques the so-called distribution method will be discussed here.

The formulation of the transportation problem is the following:

In a supply system there are m factories which must supply the needs of n warehouses. The production capacity of the factory is limited to the amounts $a_1, a_2, ..., a_m$, and the demands of the warehouses are given by $b_1, b_2, ..., b_{n.}$ The unit cost of shipping one item from factory i to warehouse j is also known, and denoted by c_{ij}. (Shipping cost is linear, which means that if x unit is shipped from factory i to warehouse j then the cost is $c_{ij} \cdot x$.) In addition, we suppose that total supply and total demand is balanced, that is $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ (the situation when this equality is not true will be later discussed). The problem is to find the minimum cost schedule that satisfies the production and demand constraints.

Let x_{ij} = the number of units shipped from factory i to warehouse j. Then the problem is:

$$c \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \to \min$$

$$\sum_{j=1}^{n} x_{ij} = a_i \quad i = 1, 2, ..., m$$

$$\sum_{i=1}^{m} x_{ij} = b_j \quad j = 1, 2, ..., n$$

$$x_{ij} \geq 0 \quad i = 1, 2, ..., m \quad j = 1, 2, ..., n$$

It was mentioned earlier that instead of a simplex algorithm the problem will be solved by the distribution method, which is based on the variables of the dual problem. For the sake of better understanding, let us take the dual of the transportation problem.

We associate a dual variable u_i with each of the first m constraints and a dual variable v_j, with each of the next n constraints. Because of the special structure of the primal problem the left-hand side of the dual constraints is a sum of type $u_i + v_j$. The relation of each constraint is of the $\leq$ type, since each primal variable is restricted to be nonnegative. There are only equalities in the primal problem, implying no restrictions for the sign of dual variables. The dual problem is:

$$\sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j \to \max$$

$$u_i + v_j \leq c_{ij} \quad i = 1, 2, ..., m \quad j = 1, 2, ..., n$$

Suppose that we are solving the transportation problem as a regular LP problem using the simplex method. When we have a feasible basic solution of the primal problem, each of the basic variables is positive. This implies the equality $u_i + v_j = c_{ij}$ for each basic variable x_{ij}, since the corresponding slackness variable must be zero in the dual problem. This creates $m+n-1$ equations in $m+n$ unknowns, which system can be solved by assigning an arbitrary value to one of the unknowns. (We usually set u_1 to zero.) After determining the values u_i and v_j we calculate the differences $\overline{c_{kl}} = c_{kl} - u_k - v_l$ for all nonbasic variables. The value $\overline{c_{kl}}$ shows to what extent the objective function would change if the variable x_{kl} entered the basis. If all $\overline{c_{kl}} \geq 0$, the current basic feasible solution is optimal. If at least one $\overline{c_{kl}} < 0$, we must select the x_{kl} which has the most negative value of $\overline{c_{kl}}$. This variable is entered in the basis, then we test the optimality again and continue the procedure until the optimum is reached.

The transportation problem always has an optimal solution as the following theorem states:

In the transportation problem the primal and dual problem both have optimal solutions, and the minimum of the primal objective function is equal to the maximum of the dual objective function.

6.1 Example of a Transportation Problem

Consider a supply system comprising three factories and three warehouses. The capacities of the factories F_1, F_2, F_3 are 20, 15, 10 units, respectively; the needs of the warehouses W_1, W_2, W_3 are 5, 20 and 20, respectively. (Total supply and total demand is equal.) Shipping costs per unit are given by the *transportation tableau* as follows:

		Warehouses			Supply
		W_1	W_2	W_3	
	F_1	0.9	1	1	20
Factories	F_2	1	1.4	0.8	15
	F_3	1.3	1	0.8	10
Demand		5	20	20	

Let x_{ij} be the number of units transported from factory F_i to warehouse W_j, $i, j = 1, 2, 3$.

Then the regular LP model of the problem is:

$$
\begin{gathered}
0.9x_{11} + x_{12} + x_{13} + x_{21} + 1.4x_{22} + 0.8x_{23} + 1.3x_{31} + x_{32} + 0.8x_{33} \to \min \\
x_{11} + x_{12} + x_{13} = 20 \\
x_{21} + x_{22} + x_{23} = 15 \\
x_{31} + x_{32} + x_{33} = 10 \\
x_{11} + x_{21} + x_{31} = 5 \\
x_{12} + x_{22} + x_{32} = 20 \\
x_{13} + x_{23} + x_{33} = 20 \\
x_{ij} \geq 0, \quad i, j = 1, 2, 3.
\end{gathered}
$$

The main points of the solution method are similar to the simplex algorithm: starting from an initial feasible solution the solution is improved step by step until the optimal solution is achieved. In order to find an initial solution we use the *northwest corner method.* This method starts by allocating as much as possible in the northwest (upper left) corner of the tableau. In our example it means that 5 units can be tied in the cell (1;1) which corresponds to factory F_1 and warehouse W_1. So 15 units left in factory F_1 and the demand of warehouse W_1 is satisfied. This means that column 1 can be removed from consideration and cell (1;2) becomes the new northwest corner. The maximum that can be allocated to this cell is 15, all that remained in F_1. As there are no more units in factory F_1, row 1 is removed from consideration and cell (2;2) becomes the new northwest corner. 5 units can be tied to this cell, then 10 units to its right-hand neighbor and finally 10 units are left for cell (3;3). The obtained initial feasible solution is the following:

	W_1	W_2	W_3	
F_1	5	15	0	20
F_2	0	5	10	15
F_3	0	0	10	10
	5	20	20	

The cells with positive amounts indicate the basic variables x_{ij} of the initial solution. We associate variables u_i to each row i and v_j to each column j. For each basic cell $(i;j)$ set the equality $u_i + v_j = c_{ij}$:

$u_1 + v_1 = 0.9; u_1 + v_2 = 1; u_2 + v_2 = 1.4; u_2 + v_3 = 0.8; u_3 + v_3 = 0.8$

Let $u_1 = 0$, and determine the values of the remaining dual variables. The result is $v_1 = 0.9; v_2 = 1; u_2 = 0.4; v_3 = 0.4$ and $u_3 = 0.4$.

We now calculate the differences $\overline{c_{kl}} = c_{kl} - u_k - v_l$ for all nonbasic cells. $\overline{c_{kl}}$ shows the change in the objective function value for a unit allocation to the nonbasic cell $(k;l)$.

$\overline{c_{13}} = c_{13} - u_1 - v_3 = 1 - 0 - 0.4 = 0.6$

$\overline{c_{21}} = c_{21} - u_2 - v_1 = 1 - 0.4 - 0.9 = -0.3$

$\overline{c_{31}} = c_{31} - u_3 - v_1 = 1.3 - 0.4 - 0.9$

$\overline{c_{32}} = c_{32} - u_3 - v_2 = 1 - 0.4 - 1 = -0.4$

The best improvement can be achieved by cell $(3;2)$, so nonbasic variable x_{32} should enter the basis. In order to identify which variable should leave the basis, we have to construct a loop that starts and ends at the cell $(3;2)$ and connects only basic cells. In our case this loop consists of the cells $(3;2) \to (2;2) \to (2;3) \to (3;3) \to (3;2)$. The amount to be allocated to the untied cell is subtracted from and added to the cells of the loop so that the availabilities and requirements remain satisfied and no values in cells can become negative. This implies that when choosing the cell to remove from the basis, we should choose the cell of the minimum value among the cells in an odd number of steps from the starting cell. In our case this minimum value is 5, and the new transportation tableau is:

	F_1	F_2	F_3	
T_1	5	15	0	20
T_2	0	0	15	15
T_3	0	5	5	10
	5	20	20	

We should test the optimality of the new solution. Calculate again the values u_i, v_j corresponding to the tied cells:

$u_1 + v_1 = 0.9; u_1 + v_2 = 1; u_2 + v_3 = 0.8; u_3 + v_2 = 1; u_3 + v_3 = 0.8$

By choosing $u_1 = 0$, we have the solution $v_1 = 0.9; v_2 = 1; u_3 = 0; v_3 = 0.8$ and $u_2 = 0$.

Calculate the values $\overline{c_{kl}}$: $\overline{c_{13}} = 1-0-0.8 = 0.2; \overline{c_{21}} = 1-0-0.9 = 0.1; \overline{c_{22}} = 1.4-0-1 = 0.4$ and $\overline{c_{31}} = 1.3 - 0 - 0.9 = 0.4$.

The nonnegativity of the obtained values implies that the last tableau contains the optimal solution of the problem. The optimum can be determined by multiplying the shipping amounts by the shipping costs: $5 \cdot 0.9 + 15 \cdot 1 + 15 \cdot 0.8 + 5 \cdot 1 + 5 \cdot 0.8 = 40.5$.

6.2 Solving a Non-standard Transportation Problem

Sometimes a transportation problem is formulated in a more general way than it was in the first subchapter. We show that after reformulating the problem the distribution method can be applied for solving it.

1. In case of maximizing (for example an upper bound is needed for the costs) the coefficients of the objective function should be multiplied by -1.

2. There can be prohibited cells in the transportation tableau (for example it is not possible to ship from a certain factory to a certain warehouse). In this case the cell in question is denoted by M, indicating an infinitely large cost. This value remains unchanged during the calculations.

3. If total supply is not equal to total demand, that is $\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$, then a fictitious factory or fictitious warehouse is introduced. Its capacity or demand is defined so as to balance total supply with total demand. All unit transportation costs to or from this fictitious location are defined to be zero (because there will be no transportation here). Sometimes fictitious transportation is not allowed in a location, for example if the demand of a given warehouse must be satisfied by all means. In this case we write M as its cost instead of 0.

Example

The need for concrete of three buildings (E_1, E_2, E_3) is supplied by three concrete factories (B_1, B_2, B_3). The buildings require 30,40,40 tons of concrete per day, respectively; the daily capacities of the factories are 60, 50 and 20 tons of concrete, respectively. The following table shows the shipping cost of one ton of concrete from each factory to each building:

	E_1	E_2	E_3	capacity
B_1	10	13	6	60
B_2	4	1	9	50
B_3	15	10	6	20
demand	30	40	40	

Find the schedule with minimal transportation cost, if it is required that factories B_1 and B_3 work with at full capacity.

Solution

As total capacity exceeds total demand, a fictitious building should be introduced with a demand of 20 tons. Because of the restrictions, all concrete produced by B_1 and B_3 must be used, so we set the transportation costs infinitely large from these factories to the fictitious building:

	E_1	E_2	E_3	E_4	capacity
B_1	10	13	6	M	60
B_2	4	1	9	0	50
B_3	15	10	6	M	20
demand	30	40	40	20	

This means that the demand of the fictitious building can only be satisfied by factory B_2 (more precisely the concrete satisfying the fictitious demand will not be produced). Taking into consideration this information we first allocate 20 tons of concrete to cell $(2; 4)$ and then start the northwest corner method for finding an initial solution. The result is:

	E_1	E_2	E_3	E_4	
B_1	30	30	0	M	60
B_2	0	10	20	20	50
B_3	0	0	20	M	20
	30	40	40	20	

Based on the tied cells we calculate the dual variables by the equations: $u_1+v_1 = 10; u_1+v_2 = 13; u_2 + v_2 = 1; u_2 + v_3 = 9; u_2 + v_4 = 0; u_3 + v_3 = 6$

By choosing $u_1 = 0$ we have: $v_1 = 10; v_2 = 13; u_2 = -12; v_3 = 21; v_4 = 12$ and $u_3 = -15$.

For the nonbasic cells:

$\overline{c_{13}} = c_{13} - u_1 - v_3 = 6 - 0 - 21 = -15$

$\overline{c_{21}} = c_{21} - u_2 - v_1 = 4 + 12 - 10 = 6$

$\overline{c_{31}} = c_{31} - u_3 - v_1 = 15 + 15 - 10 = 20$

$\overline{c_{32}} = c_{32} - u_3 - v_2 = 10 + 15 - 13 = 12$

It is worthwile to transport from factory B_1 to building E_3. The elements of the loop $(1;3) \rightarrow (2;3) \rightarrow (2;2) \rightarrow (1;2) \rightarrow (1;3)$ show that 20 tons of concrete can be reallocated to cell $(1;3)$. The tableau with the new solution is:

	E_1	E_2	E_3	E_4	
B_1	30	10	20	M	60
B_2	0	30	0	20	50
B_3	0	0	20	M	20
	30	40	40	20	

We have to calculate the dual variables corresponding to the new solution:

$u_1 + v_1 = 10; u_1 + v_2 = 13; u_1 + v_3 = 6; u_2 + v_2 = 1; u_2 + v_4 = 0; u_3 + v_3 = 6$

Let $u_1 = 0$ and we obtain: $v_1 = 10; v_2 = 13; v_3 = 6; u_2 = -12; v_4 = 12$ and $u_3 = 0$.

The values $\overline{c_{kl}}$: $\overline{c_{21}} = 4 + 12 - 10 = 6; \overline{c_{23}} = 9 + 12 - 6 = 15$; $\overline{c_{31}} = 15 - 0 - 10 = 5$ and $\overline{c_{32}} = 10 - 0 - 13 = -3$.

It means that further improvement can be achieved by entering cell $(3;2)$. The corresponding loop is: $(3;2) \rightarrow (3;3) \rightarrow (1;3) \rightarrow (1;2) \rightarrow (3;2)$, and 10 units can be tied in the new location:

	E_1	E_2	E_3	E_4	
B_1	30	0	30	M	60
B_2	0	30	0	20	50
B_3	0	10	10	M	20
	30	40	40	20	

Test the optimality of the new solution:

$u_1 + v_1 = 10; u_1 + v_3 = 6; u_2 + v_2 = 1; u_2 + v_4 = 0; u_3 + v_2 = 10; u_3 + v_3 = 6$.

The dual variables: $u_1 = 0; v_1 = 10; v_3 = 6; u_3 = 0; v_2 = 10; u_2 = -9$ and $v_4 = 9$.

$\overline{c_{12}} = 13 - 0 - 10 = 3; \overline{c_{21}} = 4 + 9 - 10 = 3; \overline{c_{23}} = 9 + 9 - 6 = 12$; $\overline{c_{31}} = 15 - 0 - 10 = 5$.

As all $\overline{c_{kl}}$ values are positive there is no more improvement in the objective function value. The optimal solution can be read from the last tableau. Factory B_2 produces only 30 tons of concrete, the others work with at full capacity. The minimal transportation cost is $30 \cdot 10 + 30 \cdot 6 + 30 \cdot 1 + 20 \cdot 0 + 10 \cdot 10 + 10 \cdot 6 = 670$.