Second midterm test of Probability Theory and Mathematical Statistics

Date: Name:
Neptun code:

THEORETICAL QUESTIONS

1. Define the 2-dimensional random vector.

2. Define the 2-dimensional discrete random vector.
What is the range and the joint distribution of such a random vector?

3. State the Cauchy–Bunyakovsky–Schwarz inequality.

4. When do we say that $\xi_1, \xi_2, \ldots, \xi_n$ is an independent sample?

5. State the Central Limit Theorem.

PROBLEMS

Please round the final results to 4 decimal places!

1.-2. The (ξ_1, ξ_2) two-dimensional discrete random vector variable is given by its joint distribution

(ξ_1, ξ_2)	1	3	5
2	0	$2p$	$3p$
4	$3p$	$2p$	0

Determine the parameter p and the value of $\mathbb{D}^2(2\xi_1 + 3\xi_2)$.

3. Let ξ be a $\mathcal{N}(\mu = 2, \sigma^2)$ distributed random variable, where σ^2 is unknown. Determine the value of σ, if we know the random variable ξ falls into the interval $(1.6, 2.4)$ with probability 0.85.

4. Let $\xi_1, \xi_2, \ldots, \xi_n$ be an $\mathrm{Exp}(\lambda)$ distributed independent sample. Give the maximal likelihood estimation of the unknown parameter λ. What is the estimation of λ using the method of moments?

5. Let us consider the 6 element sample

$$2.1,\ 6.3,\ 4.6,\ 6.4,\ 1.2,\ 3.5$$

Calculate the notable statistics, namely the sample mean $(\overline{\xi})$, the sample variance (s_n^2), the corrected sample variance (s_n^{*2}), the sample median $(\mathrm{med}(\xi))$, and the median absolute deviation $(\mathrm{MAD}(\xi))$.

SOLUTION

1.-2.

(ξ_1,ξ_2)	1	3	5
2	0	$2p$	$3p$
4	$3p$	$2p$	0

$2p+3p+3p+2p=10p=1$, thus we have that $p=0.1$.

(ξ_1,ξ_2)	1	3	5	
2	0	0.2	0.3	0.5
4	0.3	0.2	0	0.5
	0.3	0.4	0.3	

ξ_1

x_i	2	4
$p_{i\cdot}$	0.5	0.5

$$\begin{aligned}
\mathbb{E}(\xi_1) &= 2\cdot 0.5+4\cdot 0.5=3\\
\mathbb{E}(\xi_1^2) &= 2^2\cdot 0.5+4^2\cdot 0.5=10\\
\mathbb{D}^2(\xi_1) &= \mathbb{E}(\xi_1^2)-(\mathbb{E}(\xi_1))^2=10-3^2=1\\
\mathbb{D}(\xi_1) &= \sqrt{1}=1.
\end{aligned}$$

ξ_2

y_j	1	3	5
$p_{\cdot j}$	0.3	0.4	0.3

$$\begin{aligned}
\mathbb{E}(\xi_2) &= 1\cdot 0.3+3\cdot 0.4+5\cdot 0.3=3\\
\mathbb{E}(\xi_2^2) &= 1^2\cdot 0.3+3^2\cdot 0.4+5^2\cdot 0.3=3\\
\mathbb{D}^2(\xi_2) &= \mathbb{E}(\xi_2^2)-(\mathbb{E}(\xi_2))^2=11.4-3^2=2.4\\
\mathbb{D}(\xi_1) &= \sqrt{2.4}=1.5492.
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}(\xi_1,\xi_2) &= 2\cdot 3\cdot 0.2+2\cdot 5\cdot 0.3+4\cdot 1\cdot 0.3+4\cdot 3\cdot 0.2=7.8\\
\mathrm{cov}(\xi_1,\xi_2) &= \mathbb{E}(\xi_1\xi_2)-\mathbb{E}(\xi_1)\mathbb{E}(\xi_2)=7.8-3\cdot 3=-1.2\\
\mathrm{r}(\xi_1,\xi_2) &= \frac{\mathrm{cov}(\xi_1,\xi_2)}{\mathbb{D}(\xi_1)\mathbb{D}(\xi_2)}=\frac{-1.2}{1\cdot 1.5492}=-0.7746.
\end{aligned}$$

$$\begin{aligned}
\mathbb{D}^2(2\xi_1+3\xi_2) &= \mathbb{D}^2(2\xi_1)+\mathbb{D}^2(3\xi_2)+3\mathrm{cov}(2\xi_1+3\xi_2)=4\mathbb{D}^2(\xi_1)+9\mathbb{D}^2(\xi_2)+18\mathrm{cov}(\xi_1,\xi_2)=\\
&= 4\cdot 1+9\cdot 2.4+18\cdot(-1.2)=4
\end{aligned}$$

3.

$$\begin{aligned}
0.85=\mathbb{P}(1.6<\xi<2.4) &= \mathbb{P}\left(\frac{1.6-2}{\sigma}<\eta<\frac{1.6-2}{\sigma}\right)=\\
&= \mathbb{P}\left(\frac{-0.4}{\sigma}<\eta<\frac{0.4}{\sigma}\right)=\Phi\left(\frac{0.4}{\sigma}\right)-\Phi\left(-\frac{0.4}{\sigma}\right)=2\Phi\left(\frac{0.4}{\sigma}\right)-1
\end{aligned}$$

Thus we have

$$\Phi\left(\frac{0.4}{\sigma}\right)=\frac{1.85}{2}=0.925.$$

$$\frac{0.4}{\sigma}=1.44,\qquad\Rightarrow\qquad\sigma=\frac{0.4}{1.44}=0.2778.$$

4. $f(x,\lambda) = \lambda e^{-\lambda \xi_i} \Rightarrow L(\xi,\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda \xi_i}$

$$\frac{\partial}{\partial \lambda} l(\lambda) = \frac{\partial}{\partial \lambda} ln \left(\prod_{i=1}^{n} \lambda e^{-\lambda \xi_i} \right) = 0$$

$$\frac{\partial}{\partial \lambda} \sum_{i=1}^{n} (ln(\lambda) - \lambda \xi_i) = \sum_{i=1}^{n} \left(\frac{1}{\lambda} - \xi_i \right) = \frac{n}{\lambda} - \sum_{i=1}^{n} \xi_i = 0$$

Maximum-likelihood estimator:

$$\widehat{\xi} = \frac{1}{\frac{1}{n}\sum_{i=1}^{n} \xi_i} = \frac{1}{\overline{\xi}}.$$

Estimator using the method of moments

$$\mathbb{E}(\xi) = \frac{1}{\lambda} \qquad \Rightarrow \qquad \lambda = \frac{1}{\mathbb{E}(\xi)} \qquad \Rightarrow \qquad \frac{1}{\lambda} = \frac{1}{\overline{\xi}}$$

5.

$$\overline{\xi} = \frac{1}{n} \sum \xi_i = \frac{2.1 + 6.3 + 4.6 + 6.4 + 1.2 + 3.5}{6} = 4.0167$$

$$m_2 = \frac{1}{n} \sum \xi_i^2 = \frac{2.1^2 + 6.3^2 + 4.6^2 + 6.4^2 + 1.2^2 + 3.5^2}{6} = 19.985$$

$$s_n^2 = m_2 - (\overline{\xi})^2 = 19.985 - 4.0167^2 = 3.8511$$

$$s_n^{*2} = \frac{n}{n-1} s_n^2 = \frac{6}{5} \cdot 3.8511 = 4.6213.$$

The ordered sample:

$$1.2,\ 2.1,\ 3.5,\ 4.6,\ 6.3,\ 6.4$$

$\mathrm{med}(\xi) = \frac{3.5 + 4.6}{2} = 4.05$

$$|1.2 - 4.05| = 2.85$$
$$|2.1 - 4.05| = 1.95$$
$$|3.5 - 4.05| = 0.55$$
$$|4.6 - 4.05| = 0.55$$
$$|6.3 - 4.05| = 2.25$$
$$|6.4 - 4.05| = 2.35$$

$$0.55,\ 0.55,\ 1.95,\ 2.25,\ 2.35,\ 2.85$$

$\mathrm{MAD}(\xi) = \frac{1.95 + 2.25}{2} = 2.1.$