First midterm test of Probability Theory and Mathematical Statistics

Date: Name:
Neptun code:

THEORETICAL QUESTIONS

1. Define the concept of probability space.

2. An urn contains N balls, of which s are red and N-s are white. We draw n balls without replacement. Determine the probability

$$p_k = \mathbb{P}(\text{ the number of red drawn balls is } k).$$

3. Define the concept of a discrete random variable.

4. Define the expected value of a random variable ξ (both in the discrete and absolutely continuous cases).

5. List the most important properties of variance of a random variable ξ (positive semidefiniteness, quadratic homogeneity).

PROBLEMS

Please round the final results to 4 decimal places!

1. An urn contains 10 balls, of which 4 are red and 6 are white. We draw 6 balls with replacement. What is the probability that at most 2 of the balls drawn are red?

2. Given the distribution of a discrete random variable ξ with the following table:

x_i	1	2	3
p_i	0.1	0.2	0.7

Determine the expected value and variance of the random variable ξ.

3. The density function of a random variable ξ is

$$f(x) = \begin{cases} C(x^2 + 2x), & \text{when } x \in [0,1] \\ 0, & \text{otherwise.} \end{cases}$$

Determine the unknown parameter C and the expected value of the random variable $\mathbb{E}(\xi)$.

4. The probability that a customer waits more than 5 minutes at a petrol station is 0.22. Assuming the waiting time is exponential random variable, what is the probability that a customer waits less than 6 minutes?

5. The length of a workpiece is approximately a normally distributed random variable with expected value 50 mm. Determine the standard deviation of the workpiece length if the probability that the workpiece length is less than 50.05 mm is 0.85.

SOLUTION

1. Let ξ denote the number of red balls drawn. Then $\xi \sim \mathcal{B}(p = 0.4, n = 6)$. The desired probability is:

$$\begin{aligned}\mathbb{P}(\xi \leqslant 2) &= p_0 + p_1 + p_2 = \\ &= \binom{6}{0} 0.4^0 0.6^6 + \binom{6}{1} 0.4^1 0.6^5 + \binom{6}{2} 0.4^2 0.6^4 = \\ &= 0.5443.\end{aligned}$$

2. Then the random variable ξ can take the values $1, 2, 3$ with the probabilities $0.1, 0.2, 0.7$.

$$\begin{aligned}\mathbb{E}(\xi) &= 1 \cdot 0.1 + 2 \cdot 0.2 + 3 \cdot 0.7 = \frac{13}{5} = 2.6 \\ \mathbb{E}(\xi^2) &= 1^2 \cdot 0.1 + 2^2 \cdot 0.2 + 3^2 \cdot 0.7 = \frac{36}{5} = 7.2 \\ \mathbb{D}^2(\xi) &= \mathbb{E}(\xi^2) - [\mathbb{E}(\xi)]^2 = \frac{36}{5} - \left(\frac{13}{5}\right)^2 = \frac{11}{25} = 0.44\end{aligned}$$

3. Determination of the value of the normalizing constant:

$$\begin{aligned}1 = \int_{-\infty}^{\infty} f(x)dx = C \int_0^1 (x^2 + 2x)dx = C \left[\frac{x^3}{3} + x^2\right]_{x=0}^{x=1} &= \\ = C\left(\frac{1}{3} + 1\right) = \frac{4}{3}C \quad &\Longrightarrow \\ C = \frac{3}{4}.&\end{aligned}$$

Determination of the expected value:

$$\begin{aligned}\mathbb{E}(\xi) &= \int_{-\infty}^{\infty} xf(x)dx = \frac{3}{4}\int_0^1 x(x^2 + 2x)dx = \frac{3}{4}\int_0^1 (x^3 + 2x^2)dx = \\ &= \frac{3}{4}\left[\frac{x^4}{4} + \frac{2x^3}{3}\right]_{x=0}^{x=1} = \frac{3}{4}\left(\frac{1}{4} + \frac{2}{3}\right) = \\ &= \frac{11}{16} = 0.6875.\end{aligned}$$

4. Let η denote the waiting time measured in minutes. Since $\eta \sim \text{Exp}(\lambda)$, we have that

$$F_\eta(x) = 1 - e^{-\lambda x}, \quad (x > 0).$$

Determination of the parameter λ:

$$0.22 = \mathbb{P}(\eta > 5) = 1 - \mathbb{P}(\eta < 5) = 1 - F_\eta(5) = 1 - (1 - e^{-\lambda \cdot 5}) = e^{-5\lambda},$$

from which we get that $-5\lambda = \ln(0.22)$, that is,

$$\lambda = \frac{\ln(0.22)}{-5}.$$

The desired probability:

$$\mathbb{P}(\eta < 6) = F_\eta(6) = 1 - e^{-\lambda \cdot 6} = 1 - e^{\frac{6 \cdot \ln(0.22)}{5}} = 0.8375.$$

5. Let ξ denote the length of the workpiece. We know that $\xi \sim \mathcal{N}(m = 50, \sigma^2)$.

First we have to determine the value of the unknown standard deviation. We have to apply the "standardization".

$$0.85 = \mathbb{P}(\xi < 50.05) = \mathbb{P}\left(\eta < \frac{50.05 - 50}{\sigma}\right) = \Phi\left(\frac{0.05}{\sigma}\right),$$

from which the unknown parameter σ can be easily expressed:

$$1.04 = \frac{0.05}{\sigma}, \quad \sigma = \frac{0.05}{1.04} = 0.0481.$$