Probability and Mathematical Statistics

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Seminar 8

Absolutely continuous random vector variables,
Multidimensional random vector variables

1. Absolutely continuous random vector variables

Independence and uncorrelatedness

Problem 1

The joint density function of the random vector (ξ, η):

$$f(x,y) = \begin{cases} \frac{1}{4}\left(1 + x^3 y - x y^3\right), & \text{ha } -1 \leqslant x \leqslant 1, -1 \leqslant y \leqslant 1; \\ 0, & \text{otherwise} . \end{cases}$$

Let us examine the marginal distributions in terms of independence and uncorrelatedness.

Solution to Problem 1

- **Marginal distributions:**

$$f_\xi(x) = \int_{-\infty}^{+\infty} f(x,y)dy = \frac{1}{4}\int_{-1}^{1}(1+x^3y-xy^3)dy =$$

$$= \frac{1}{4}\left[y + x^3\frac{y^2}{2} - x\frac{y^4}{4}\right]_{y=-1}^{y=1} = \frac{1}{2},$$

if $x \in [-1,1]$, we obtain that

$$f_\xi(x) = \begin{cases} \frac{1}{2}, & \text{ha } x \in]-1,1[; \\ 0, & \text{egyébként}, \end{cases}$$

thus, we obtain that $\xi \sim U[-1,1]$, $\eta \sim U[-1,1]$

Solution to Problem 1, continuation

- **Determination of the covariance**

$$\mathbb{E}(\xi\eta) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xyf(x,y)dydx = \frac{1}{4}\int_{-1}^{+1}\int_{-1}^{+1} xy(1+x^3y-xy^3)dydx =$$

$$= \frac{1}{4}\int_{-1}^{+1}\int_{-1}^{+1}(xy+x^4y^2-x^2y^4)dydx =$$

$$= \frac{1}{4}\int_{-1}^{+1}\left[x\frac{y^2}{2}+x^4\frac{y^3}{3}-x^2\frac{y^5}{5}\right]_{y=-1}^{y=1} dx = \frac{1}{4}\int_{-1}^{+1}\left(\frac{2}{3}x^4-\frac{2}{5}x^2\right)dx =$$

$$= \frac{1}{4}\left[\frac{2}{3}\frac{x^5}{5}-\frac{2}{5}\frac{x^3}{3}\right]_{x=-1}^{x=1} = \frac{2}{3}\cdot\frac{2}{5}-\frac{2}{5}\cdot\frac{2}{3} = 0.$$

Thus

$$\operatorname{cov}(\xi,\eta) = \mathbb{E}(\xi\eta) - \mathbb{E}(\xi)\mathbb{E}(\eta) = 0 - 0\cdot 0 = 0.$$

- **Uncorrelatedness and independence** It can be seen that ξ and η are uncorrelated but not independent.

Problem 2

Determine, for the random vector (ξ, η), the normalization constant (that is, the value of the parameter C), the marginal distribution functions and expected values, as well as the value of $\operatorname{cov}(\xi, \eta)$, if the joint density function of the vector (ξ, η) is given by

b. •

$$f(x,y) = \begin{cases} C(xy+1), & \text{ha } 0 < x < 1, 0 < y < 2; \\ 0, & \text{otherwise }. \end{cases}$$

c. •

$$f(x,y) = \begin{cases} C(xy+x), & \text{ha } 0 < x < 1, 0 < y < 2; \\ 0, & \text{otherwise }. \end{cases}$$

Solution to Problem 2. b

Let be $T = [0,1] \times [0,2]$. $(f(x,y) = C(xy+1), (x,y \in T))$.

- **The value of the constant C:**

$$1 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y)dydx = C \int_0^1 \left(\int_0^2 (xy+1)dy \right) dx =$$

$$= C \int_0^1 \left[\frac{xy^2}{2} + y \right]_{y=0}^{y=2} dx =$$

$$= C \int_0^1 (2x+2)dx = C \left[x^2 + 2x\right]_{x=0}^{x=1} = 3C \qquad \Longrightarrow \qquad C = \frac{1}{3}.$$

- **The marginal distribution functions:**

$$f_1(x) = \int_{-\infty}^{+\infty} f(x,y)dy = \frac{1}{3} \int_0^2 (xy+1)dy = \frac{1}{3} \left[\frac{xy^2}{2} + y \right]_{x=0}^{x=2} =$$

$$= \frac{2}{3}(x+1) \qquad (x \in [0,1]),$$

$$f_2(y) = \int_{-\infty}^{+\infty} f(x,y)dx = \frac{1}{3} \int_0^2 (xy+1)dx = \frac{1}{3} \left[\frac{x^2y}{2} + x \right]_{x=0}^{x=1} =$$

$$= \frac{1}{3} \left(\frac{1}{2}y + 1 \right) = \frac{1}{6}(y+2) \qquad (y \in [0,2]).$$

- **The expected values of the marginals:**

$$\mathbb{E}(\xi) = \int_{-\infty}^{+\infty} x f_1(x)dx = \frac{2}{3}\int_0^1 x(x+1)dx = \frac{2}{3}\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_{x=0}^{x=1} =$$
$$= \frac{2}{3}\left(\frac{1}{3} + \frac{1}{2}\right) = \frac{5}{9};$$
$$\mathbb{E}(\eta) = \int_{-\infty}^{+\infty} y f_2(y)dy = \frac{1}{6}\int_0^2 y(y+2)dy = \frac{1}{6}\int_0^2 (y^2+2y)dy =$$
$$= \frac{1}{6}\left[\frac{y^3}{3} + y^2\right]_{x=0}^{x=2} = \frac{1}{6}\left(\frac{1}{3}+4\right) = \frac{13}{18}.$$

- **The value of the covariance:**

$$\mathbb{E}(\xi\cdot\eta) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xyf(x,y)dydx = \frac{1}{3}\int_0^1\left(\int_0^2 xy(xy+1)dy\right)dx =$$
$$= \frac{1}{3}\int_0^1\left(\int_0^2 (x^2y^2+xy)dy\right)dx = \frac{1}{3}\int_0^1\left[\frac{x^2y^3}{3} + \frac{xy^2}{2}\right]_{x=0}^{x=2} dx =$$
$$= \frac{1}{3}\int_0^1\left(\frac{8}{3}x^2 + 2x\right)dx = \frac{1}{3}\left[\frac{8}{3}\frac{x^3}{3} + x^2\right]_{x=0}^{x=1} = \frac{1}{3}\left(\frac{8}{9}+1\right) = \frac{17}{27};$$
$$\operatorname{cov}(\xi,\eta) = \mathbb{E}(\xi\cdot\eta) - \mathbb{E}(\xi)\mathbb{E}(\eta) = \frac{17}{27} - \frac{5}{9}\cdot\frac{19}{18} = \frac{37}{162} = 0.2284.$$

Solution to Problem 2. c

$(f(x,y) = xy + x, \ (x,y) \in T)$.
Since $f(x,y) = (C_1 x)(C_2(y+1))$ and the support of the joint distribution function (the set where it is nonzero) is a rectangle, ξ and η are independent.

- **The marginal distribution functions:**

$$f_1(x) = \begin{cases} C_1 x, & \text{ha } x \in [0,1]; \\ 0, & \text{otherwise }. \end{cases}$$

$$f_2(y) = \begin{cases} C_2(y+1), & \text{ha } y \in [0,2]; \\ 0, & \text{otherwise }. \end{cases}$$

$$1 = \int_{-\infty}^{+\infty} f_1(x)dx = C_1 \int_0^1 x dx = C_1 \left[\frac{x^2}{2}\right]_{x=0}^{x=1} = \frac{1}{2} C_1,$$

Thus $C_1 = 2$.

$$1 = \int_{-\infty}^{+\infty} f_2(y)dy = C_2 \int_0^2 (y+2)dy = C_2 \left[\frac{y^2}{2} + 2y\right]_{x=0}^{x=2} = 6C_2,$$

thus $C_2 = \frac{1}{6}$.

Solution to Problem 2. c, continuation

- **The expected values of the marginals:**

$$\mathbb{E}(\xi) = \int_{-\infty}^{+\infty} x f_1(x) dx = 2\int_0^1 x \cdot x dx = 2\left[\frac{x^3}{3}\right]_{x=0}^{x=1} = \frac{2}{3};$$

$$\mathbb{E}(\eta) = \int_{-\infty}^{+\infty} y f_2(y) dy = \frac{1}{6}\int_0^2 y(y+1) dy = \frac{1}{6}\int_0^2 (y^2+y) dy =$$

$$= \frac{1}{6}\left[\frac{y^3}{3} + \frac{y^2}{2}\right]_{x=0}^{x=2} = \frac{1}{6}\left(\frac{1}{3} + \frac{1}{2}\right) = \frac{5}{36}.$$

- **The covariance:** $\mathrm{cov}(\xi, \eta) = 0$, since ξ and η are independent, and thus they are uncorrelated.

Problem 3

Determine the value of the constant k and the marginal distributions if the joint density function is

$$f(x,y) = \begin{cases} Cxye^{-k(x^2+y^2)}, \text{ ha } x > 0, y > 0; \\ 0, \text{ otherwise }. \end{cases}$$

Solution to Problem 3

The marginal distributions of ξ and η are identically distributed and independent, since

$$f(x,y) = f_1(x)f_2(y) = \left(xe^{-kx^2}\right)\left(ye^{-ky^2}\right) \qquad (x>0, y>0),$$

and the support of the joint distribution function is a rectangle.

- **Determination of the value of the constant C**

$$1 = \int_{-\infty}^{+\infty} xe^{-kx^2}\,dx = -\frac{1}{2k}\int_0^{+\infty}(-2kx)e^{-kx^2}\,dx =$$

$$= -\frac{1}{2k}\left[e^{-kx^2}\right]_{x=0}^{x=\infty} = -\frac{1}{2k}\cdot(-1) = \frac{1}{2k} \quad \Longleftrightarrow \quad k = \frac{1}{2}.$$

- **Marginal distributions:**

$$f_1(x) = \begin{cases} xe^{-\frac{x^2}{2}}, & \text{if } x>0; \\ 0, & \text{otherwise }. \end{cases}$$

$$f(x,y) = \begin{cases} ye^{-\frac{y^2}{2}}, & \text{if } y>0; \\ 0, & \text{otherwise }. \end{cases}$$

Problem 4

Determine the value of the parameter C, the probability $\mathbb{P}(\xi < 1, \eta > 1)$, and the value of $\mathrm{cov}(\xi, \eta)$, if the joint density function of the random vector (ξ, η) is given by

a. •

$$f(x,y) = \begin{cases} C\left(x + \frac{y}{2}\right), & \text{ha } 0 < x < 1, 0 < y < 2; \\ 0, & \text{otherwise }. \end{cases}$$

Solution to Problem 4

The joint density function for part a.:

$$f_B(x,y) = \begin{cases} C_B\left(x+\frac{y}{B}\right), & \text{ha } x\in[0,1], y\in[0,B]; \\ 0, & \text{otherwise }. \end{cases}$$

- **Determination of the value of the constant C_B**

$$1 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)dydx = C_B\int_0^1\left(\int_0^B\left(x+\frac{y}{B}\right)dy\right)dx =$$

$$= C_B\int_0^1\left[xy+\frac{y^2}{2B}\right]_{y=0}^{y=B}dx = C_B\int_0^1(Bx+\frac{B}{2})dx =$$

$$= C_B\left[\frac{Bx^2}{2}+\frac{Bx}{2}\right]_{x=0}^{x=1} = BC_B \qquad\Longrightarrow\qquad C_B=\frac{1}{B}.$$

Solution to Problem 4, continuation

- Determination of the expected values of the marginals:

$$\mathbb{E}(\xi_B) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x f_B(x,y) dy dx = \frac{1}{B} \int_0^1 \left(\int_0^B x \left(x + \frac{y}{B} \right) dy \right) dx =$$

$$= \frac{1}{B} \int_0^1 \left(\int_0^B \left(x^2 + \frac{xy}{B} \right) dy \right) dx = \frac{1}{B} \int_0^1 \left[x^2 y + \frac{xy^2}{B} \right]_{y=0}^{y=B} dx =$$

$$= \frac{1}{B} \int_0^1 \left(Bx^2 + \frac{B}{2} x \right) dx = \int_0^1 \left(x^2 + \frac{x}{2} \right) dx = \left[\frac{x^3}{3} + x^2 \right]_{x=0}^{x=1} =$$

$$= \frac{1}{3} + 1 = \frac{4}{3};$$

$$\mathbb{E}(\eta_B) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} y f_B(x,y) dy dx = \frac{1}{B} \int_0^1 \left(\int_0^B y \left(x + \frac{y}{B} \right) dy \right) dx =$$

$$= \frac{1}{B} \int_0^1 \left(\int_0^B \left(xy + \frac{y^2}{B} \right) dy \right) dx = \frac{1}{B} \int_0^1 \left[\frac{xy^2}{2} + \frac{y^3}{3B} \right]_{y=0}^{y=B} dx =$$

$$= \frac{1}{B} \int_0^1 \left(\frac{B^2 x}{2} + \frac{B^2}{3} x \right) dx = B \int_0^1 \left(\frac{x}{2} + \frac{1}{3} \right) dx =$$

$$= B \left[\frac{x^2}{4} + \frac{x}{3} \right]_{x=0}^{x=1} = \frac{7}{12} B.$$

Solution to Problem 4, continuation

- The value of the covariance:

$$\mathbb{E}(\xi_B \cdot \eta_B) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} xyf(x,y)dydx =$$
$$= \frac{1}{B} \int_0^1 \left(\int_0^B xy \left(x + \frac{y}{B} \right) dy \right) dx =$$
$$= \frac{1}{B} \int_0^1 \left(\int_0^B \left(x^2 y + \frac{xy^2}{B} \right) dy \right) dx = \frac{1}{B} \int_0^1 \left[\frac{x^2 y^2}{2} + \frac{xy^3}{3B} \right]_{x=0}^{x=B} dx =$$
$$= \frac{1}{B} \int_0^1 \left(\frac{B^2 x^2}{2} + \frac{B^2}{3} \right) dx = B \int_0^1 \left(\frac{x^2}{2} + \frac{1}{3} \right) dx =$$
$$= B \left[\frac{x^3}{3} + \frac{x}{3} \right]_{x=0}^{x=1} = B \left(\frac{1}{6} + \frac{1}{3} \right) = \frac{1}{2} B;$$
$$\operatorname{cov}(\xi_B, \eta_B) = \mathbb{E}(\xi_B \cdot \eta_B) - \mathbb{E}(\xi_B)\mathbb{E}(\eta_B) = \frac{1}{2} B - \frac{4}{3} \cdot \frac{7}{12} B = -\frac{5}{18} B.$$

2. Multidimensional Uniform Distribution

Problem 5

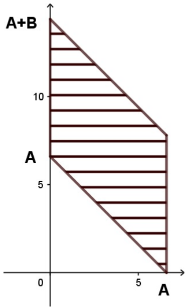
The joint density function of the random vector variable (ξ, η) is:

$$f(x,y) = \begin{cases} C, & \text{ha } 0 < x < 6.6, 6.6 - x < y < 14.4 - x; \\ 0, & \text{egyébként} \ . \end{cases}$$

Determine the value of the constant C, the marginal distributions, and the value of $\mathrm{cov}(\xi, \eta)$.

Solution to Problem 5

Solve the problem parametrically. The values of the parameters A and B are: $A = 6.6$, $B = 7.8$.



- **The value of the constant C:** $C = \frac{1}{T}$, where T is the area of the parallelogram. Thus, we obtain that

$$C = \frac{1}{AB} = \frac{1}{6.6 \cdot 7.8} = \frac{1}{51.48} = 0.0194.$$

Solution to Problem 5, continuation

- **The expected values of the marginals:** are the coordinates $(\mathbb{E}(\xi), \mathbb{E}(\eta))$ of the center of the paralelogram. This can be easily obtained as the midpoint of the segment with endpoints $(0, A+B)$ and $(A, 0)$. Thus we obtain that

$$\mathbb{E}(\xi) = \frac{0+A}{2} = \frac{A}{2} = 3.3, \quad \mathbb{E}(\eta) = \frac{A+B+0}{2} = \frac{A+B}{2} = 7.2.$$

- **The value of** $\mathrm{cov}(\xi, \eta)$**:**

$$\mathbb{E}(\xi\eta) = \frac{1}{AB} \int_0^A \left(\int_{-x+A}^{-x+A+B} xy dy \right) dx = \frac{1}{2AB} \int_0^A \left[xy^2 \right]_{y=-x+A}^{y=-x+A+B} dx.$$

$$\begin{aligned} \left[xy^2 \right]_{y=-x+A}^{y=-x+A+B} &= x(-x+A+B)^2 - (-x+A)^2 = \\ &= x(-x+A+B-x+A)(-x+A+B+x-A) = \\ &= x(-2x+2A+B)B. \end{aligned}$$

Solution to Problem 5, continuation

Thus, we obtain that

$$\begin{aligned}
\mathbb{E}(\xi\eta) &= \frac{1}{2AB} \cdot B \int_0^A (-2x^2 + (2A+B)x)dx = \\
&= \frac{1}{2A}\left[\frac{-2x^3}{3} + \frac{(2A+B)x^2}{2}\right]_{x=0}^{x=A} = \frac{-A^2}{3} + \frac{(2A+B)A}{4} = \\
&= \frac{1}{12}(-4A^2 + 6A^2 + 3AB) = \frac{1}{12}(2A^2 + 3AB) = \\
&= 20.13, \\
\operatorname{cov}(\xi,\eta) &= \mathbb{E}(\xi\cdot\eta) - \mathbb{E}(\xi)\mathbb{E}(\eta) = \frac{1}{12}(2A^2+3AB) - \frac{1}{4}(A^2+AB) = \\
&= \frac{1}{12}(2A^2 + 3AB - 3A^2 - 3AB) = -\frac{1}{12}A^2 = -3.63.
\end{aligned}$$

3. Multidimensional Normal Distribution

Problem 6

Determine the unknown parameters $(m_1,\ m_2,\ \sigma_1,\ \sigma_2,\ r)$ of the normally distributed random vector (ξ, η), which is given by its joint density function.

$$f(x,y) = \frac{1}{4\pi} e^{-\frac{4x^2+1-2y+y^2}{8}} \qquad ((x,y) \in \mathbb{R}^2).$$

Solution to Problem 6

In this problem, we can use that the random variables ξ and η are independent. The unknown parameters are: m_1, m_2, σ_1, σ_2, r.
Since

$$f(x,y) = \frac{1}{4\pi} e^{-\frac{4x^2+1-2y+y^2}{8}} = \left(C_1 e^{-\frac{x^2}{2}} \right) \left(C_2 e^{-\frac{(y-1)^2}{8}} \right),$$

Thus, $\xi \sim \mathcal{N}(m_1 = 0, \sigma_1^2 = 1)$ and $\eta \sim \mathcal{N}(m_2 = 1, \sigma_2^2 = 4)$ are independent random variables.
Checking the constant

$$C_1 C_2 = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2\pi} \cdot 2} = \frac{1}{4\pi},$$

and $\mathrm{r} = 0$.

Problem 7

Determine the unknown parameters (m_1, m_2, σ_1, σ_2, r) of the normally distributed random vector (ξ, η), given its joint density function for any $(x, y) \in \mathbb{R}^2$.

$$f(x, y) = \frac{1}{2\pi} e^{\frac{1}{2\sqrt{2}\pi} e^{-\frac{x^2 - 2xy + 3y^2}{4}}}.$$

Solution to Problem 7

It is easy to see that $m_1 = m_2 = 0.$. To determine the distributions of marginals in the joint density function we must convert the exponent by completing the square.

- The distribution of ξ: Rewrite the exponent by completing the square with respect to the variable y.

$$-\frac{x^2 - 2xy + 3y^2}{4} = -\frac{3}{4}\left(y^2 - \frac{2}{3}xy + \frac{1}{3}x^2\right) =$$

$$= -\frac{3}{4}\left(\left(y - \frac{1}{3}x\right)^2 - \frac{1}{9}x^2 + \frac{1}{3}x^2\right) =$$

$$= -\frac{3}{4}\left(y - \frac{1}{3}x\right)^2 - \frac{1}{6}x^2 = -\frac{\left(y - \frac{1}{3}x\right)^2}{2\left(\sqrt{\frac{2}{3}}\right)^2} - \frac{1}{6}x^2,$$

Solution to Problem 7, continuation

thus we obtain, that

$$f_\xi(x) = \frac{\sqrt{2}}{4\pi}\int_{-\infty}^{+\infty} e^{\frac{-\left(y-\frac{1}{3}x\right)^2}{2\left(\sqrt{\frac{2}{3}}\right)^2}} e^{-\frac{1}{6}x^2}dy = \frac{\sqrt{2}}{4\pi}e^{-\frac{1}{6}x^2}\underbrace{\int_{-\infty}^{+\infty} e^{-\frac{\left(y-\frac{1}{3}x\right)^2}{2\left(\sqrt{\frac{2}{3}}\right)^2}}dx}_{I_1=\sqrt{2\pi}\cdot\sqrt{\frac{2}{3}}} =$$

$$= \frac{\sqrt{2}}{4\pi}\sqrt{2\pi}\sqrt{\frac{2}{3}}e^{-\frac{1}{6}x^2} = \frac{1}{\sqrt{2\pi}\sqrt{3}}e^{-\frac{1}{6}x^2} \qquad (x \in \mathbb{R}).$$

It is easy to see that $I_1 = \sqrt{2\pi}\sqrt{\frac{2}{3}}$, because the integrand of integral I_1 is a density function (without normalized constant) distributed $\mathcal{N}\left(m = \frac{1}{3}x, \sigma^2 = \left(\sqrt{\frac{2}{3}}\right)^2\right)$, from which we obtain that $\xi \sim \mathcal{N}(m_1 = 0, \sigma_1^2 = (\sqrt{3})^2)$.

Solution to Problem 7, continuation

- The distribution of η: Completing the square in the exponent with respect to the variable x.

$$-\frac{x^2-2xy+3y^2}{4} = -\frac{1}{4}\left((x-y)^2-y^2+3y^2\right) = -\frac{1}{4}(x-y)^2-\frac{1}{2}y^2 = \\ = -\frac{(x-y)^2}{2(\sqrt{2})^2}-\frac{y^2}{2}$$

thus we obtain that

$$f_\eta(y) = \frac{\sqrt{2}}{4\pi}\int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{2(\sqrt{2})^2}} e^{-\frac{y^2}{2}}\,dx = \frac{\sqrt{2}}{4\pi} e^{-\frac{y^2}{2}} \underbrace{\int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{2(\sqrt{2})^2}dx}}_{I_2=\sqrt{2\pi}\sqrt{2}=2\sqrt{\pi}} = \\ = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}.$$

It is easy to see that $I_2 = 2\sqrt{\pi}$, because the integrand of integral I_1 is a density function (without normalized constant) distributed $\mathcal{N}(m=y, \sigma^2=(\sqrt{2})^2)$, from which we obtain that $\eta \sim \mathcal{N}(m_2=0, \sigma_2^2=1)$, that is, it is a standard normal distributed random variable.

Solution to Problem 7, continuation

- Calculation of $\operatorname{cov}(\xi,\eta)$: Since $\mathbb{E}(\xi)=\mathbb{E}(\eta)=0$, thus $\operatorname{cov}(\xi,\eta)=\mathbb{E}(\xi\eta)$.

$$\operatorname{cov}(\xi,\eta)=\mathbb{E}(\xi\eta)=\frac{\sqrt{2}}{4\pi}\int_{-\infty}^{+\infty} ye^{-\frac{y^2}{2}}\underbrace{\left(\int_{-\infty}^{+\infty} xe^{-\frac{(x-y)^2}{2(\sqrt{2})^2}}\,dx\right)}_{I_3=\sqrt{2\pi}\sqrt{2}y=2\sqrt{\pi}y}dy=$$

$$=\frac{\sqrt{2}}{4\pi}2\sqrt{\pi}\underbrace{\int_{-\infty}^{+\infty} y^2e^{-\frac{y^2}{2}}}_{I_4=\sqrt{2\pi}}=\frac{\sqrt{2}}{4\pi}2\sqrt{\pi}\sqrt{2\pi}=1.$$

The value $I_3=2\sqrt{\pi}y$ comes from the fact that the integrand (with the appropriate constant factor) gives the expected value of a $\mathcal{N}(m=y,\sigma^2=(\sqrt{2})^2)$ distributed random variable. And $I_4=\sqrt{2\pi}$ follows from the fact that the integrand (with the appropriate constant factor) is the second moment of a standard normal distributed random variable.

Solution to Problem 7, continuation

- The value of $\mathrm{r}(\xi,\eta)$:

$$\mathrm{r}=\mathrm{r}(\xi,\eta)=\frac{\mathrm{cov}(\xi,\eta)}{\mathbb{D}^2(\xi)\mathbb{D}^2(\eta)}=\frac{\mathbb{E}(\xi\cdot\eta)-\mathbb{E}(\xi)\mathbb{E}(\eta)}{\mathbb{D}^2(\xi)\mathbb{D}^2(\eta)}=\frac{1-0\cdot 0}{\sqrt{3}\cdot 1}=\frac{1}{\sqrt{3}}.$$

- Checking the constant:

$$\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\mathrm{r}^2}}=\frac{1}{2\cdot\pi\cdot\sqrt{3}\cdot 1\cdot\sqrt{1-\frac{1}{3}}}=\frac{1}{2\sqrt{2}\pi}=\frac{\sqrt{2}}{4\pi}.$$

- Checking the exponent:

$$-\frac{1}{2(1-r^2)}\left[\frac{(x_1-m_1)^2}{\sigma_1^2}+2r\frac{(x-m_1)(x-m_2)}{\sigma_1\sigma_2}+\frac{(y-m_2)^2}{\sigma_2^2}\right]=$$

$$=-\frac{1}{2\left(1-\frac{1}{3}\right)}\left[\frac{x^2}{3}-\frac{2}{\sqrt{3}}\frac{xy}{\sqrt{3}}+\frac{y^2}{1}\right]=-\frac{1}{4}\left(x^2-2xy+3y^2\right).$$

Problem 8

Determine the unknown parameters $(m_1,\ m_2,\ \sigma_1^2,\ \sigma_2^2,\ r)$ of the (ξ, η) normally distributed random vector, given by its joint density function.

$$f(x,y) = \frac{1}{2\sqrt{2}\pi} e^{-\frac{1}{4}(x^2+3y^2-2xy+8x-16y+24)} \qquad ((x,y) \in \mathbb{R}^2).$$

Solution to Problem 8

In this problem, we have the (ξ, η) two-dimensional normally distributed random variables whose marginal distributions have non-zero expected values.

- **Determining the expected values of the marginal distributions:** We set the partial derivative equal to 0 and then solve the resulting system of equations.

$$\left.\begin{aligned} \frac{\partial f(x,y)}{\partial x} &= f(x,y)\left(-\frac{1}{4}\right)(2x+2y+8) = 0 \\ \frac{\partial f(x,y)}{\partial y} &= f(x,y)\left(-\frac{1}{4}\right)(6y-2x-16) = 0 \end{aligned}\right\},$$

thus we obtain that

$$\left.\begin{aligned} x - y &= -4 \\ x - 3y &= -8 \end{aligned}\right\} \implies \left.\begin{aligned} x &= -2 \\ y &= 2 \end{aligned}\right\} \implies m_1 = -2, m_2 = 2.$$

Solution to Problem 8, continuation

- **Shifting to the origin:** Since the translation leaves the variances and the correlation coefficient unchanged, we can apply a shift that makes the marginals have zero expected values.
  In the joint density function, the exponent changes as follows:

$$-\frac{1}{4}\left((x-2)^2+3(y+2)^2-2(x-2)(y+2)+8(x-2)-16(y+2)+24\right)=$$
$$=-\frac{1}{4}(x^2-2xy+3y^2).$$

  That is, the resulting joint density function is:

$$f(x-2,y+2)=\frac{1}{2\sqrt{2}\pi}e^{-\frac{x^2-2xy+3y^2}{4}}, \qquad ((x,y)\in\mathbb{R}^2).$$

  Thus, from Problem 7 we obtain that

$$\sigma_1=\sqrt{3}, \qquad \sigma_2=1, \qquad \mathrm{r}=\frac{1}{\sqrt{3}}.$$

End of Seminar 8