

Probability and Mathematical Statistics

Miskolc, 2025.

Dr. Tamás Glavosits

Seminar 7

Discrete random vector variables

1. Discrete random vector variables

Problem 1

We roll two dice, one of them is red and the other one is blue. Let ξ_1 and ξ_2 denote the numbers obtained on the two dice.

Determine:

- The distribution, expected value, and variance of $\xi = \min(\xi_1, \xi_2)$;
- The distribution, expected value, and variance of $\eta = \max(\xi_1, \xi_2)$;
- The joint distribution of (ξ, η) , as well as the values of $\text{cov}(\xi, \eta)$ and $r(\xi, \eta)$.

Solution to Problem 1.a

a. The distribution of $\min(\xi_1, \xi_2)$:

$$\begin{aligned}\mathbb{P}(\min(\xi_1, \xi_2) = 1) &= \mathbb{P}((1, 1), (1, 2), \dots, (1, 6), (2, 1), (3, 1), \dots, (6, 1)) = \\ &= \frac{2 \cdot 6 - 1}{36} = \frac{11}{36},\end{aligned}$$

$$\begin{aligned}\mathbb{P}(\min(\xi_1, \xi_2) = 2) &= \mathbb{P}((2, 2), (2, 3), \dots, (2, 6), (3, 2), (4, 2), \dots, (6, 2)) = \\ &= \frac{2 \cdot 5 - 1}{36} = \frac{9}{36},\end{aligned}$$

(...)

Similarly, we obtain that

$$\mathbb{P}(\min(\xi_1, \xi_2) = k) = \frac{2(6 - k + 1) - 1}{36} \quad (k = 1, 2, \dots, 6).$$

The expected value and variance of $\min(\xi_1, \xi_2)$:

x_i	1	2	3	4	5	6
p_i	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$

Solution to Problem 1. a, continuation

$$\begin{aligned}\mathbb{E}(\xi) &= \sum_i x_i p_i = 1 \cdot \frac{11}{36} + 2 \cdot \frac{9}{36} + 3 \cdot \frac{7}{36} + 4 \cdot \frac{5}{36} + 5 \cdot \frac{3}{36} + 6 \cdot \frac{1}{36} = \\ &= \frac{91}{36} = 2.5278,\end{aligned}$$

$$\begin{aligned}\mathbb{E}(\xi^2) &= \sum_i x_i^2 p_i = 1^2 \cdot \frac{11}{36} + 2^2 \cdot \frac{9}{36} + 3^2 \cdot \frac{7}{36} + 4^2 \cdot \frac{5}{36} + 5^2 \cdot \frac{3}{36} + 6^2 \cdot \frac{1}{36} = \\ &= \frac{301}{36} = 8.3611,\end{aligned}$$

$$\mathbb{D}^2(\xi) = \mathbb{E}(\xi^2) - [\mathbb{E}(\xi)]^2 = \frac{301}{36} - \left(\frac{91}{36}\right)^2 = 1.9715.$$

Solution to Problem 1. b

b. The distribution of $\max(\xi_1, \xi_2)$:

$$\begin{aligned}\mathbb{P}(\max(\xi_1, \xi_2) = 1) &= \mathbb{P}((6, 1), (6, 2), \dots, (6, 6), (1, 6), (2, 6), \dots, (5, 6)) = \\ &= \frac{2 \cdot 6 - 1}{36} = \frac{11}{36},\end{aligned}$$

$$\begin{aligned}\mathbb{P}(\max(\xi_1, \xi_2) = 2) &= \mathbb{P}((5, 1), (5, 2), \dots, (5, 5), (1, 5), (2, 5), \dots, (4, 5)) = \\ &= \frac{2 \cdot 5 - 1}{36} = \frac{9}{36},\end{aligned}$$

(...)

Similarly, we obtain that

$$\mathbb{P}(\min(\xi_1, \xi_2) = k) = \frac{2k - 1}{36} \quad (k = 1, 2, \dots, 6).$$

The expected value and variance of $\min(\xi_1, \xi_2)$:

x_i	1	2	3	4	5	6
p_i	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

Solution to Problem 1. b, continuation

$$\begin{aligned}\mathbb{E}(\xi) &= \sum_i x_i p_i = 1 \cdot \frac{1}{36} + 2 \cdot \frac{3}{36} + 3 \cdot \frac{5}{36} + 4 \cdot \frac{7}{36} + 5 \cdot \frac{9}{36} + 6 \cdot \frac{11}{36} = \\ &= \frac{161}{36} = 4.4722,\end{aligned}$$

$$\begin{aligned}\mathbb{E}(\xi^2) &= \sum_i x_i^2 p_i = 1^2 \cdot \frac{1}{36} + 2^2 \cdot \frac{3}{36} + 3^2 \cdot \frac{5}{36} + 4^2 \cdot \frac{7}{36} + 5^2 \cdot \frac{9}{36} + 6^2 \cdot \frac{11}{36} = \\ &= \frac{791}{36} = 21.9722,\end{aligned}$$

$$\mathbb{D}^2(\xi) = \mathbb{E}(\xi^2) - [\mathbb{E}(\xi)]^2 = \frac{791}{36} - \left(\frac{161}{36}\right)^2 = 1.9715.$$

Solution to Problem 1. c

c. The joint distribution of (ξ, η) , where $\xi = \min(\xi_1, \xi_2)$ and $\eta = \max(\xi_1, \xi_2)$. It is worth noting that:

- if $i < j$, then $p_{ij} = \frac{2}{36}$,
- if $i = j$, then $p_{ij} = \frac{1}{36}$,
- if $i > j$, then $p_{ij} = 0$.

Thus, it is easy to fill in the joint distribution table.

(ξ, η)	1	2	3	4	5	6
1	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$
2	0	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$
3	0	0	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$
4	0	0	0	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$
5	0	0	0	0	$\frac{1}{36}$	$\frac{2}{36}$
6	0	0	0	0	0	$\frac{1}{36}$

Solution to Problem 1. c, continuation

$$\begin{aligned}\mathbb{E}(\xi \cdot \eta) &= \sum_{ij} x_i y_j p_{ij} = \sum_{i=1}^6 i^2 \frac{1}{36} + \sum_{1 \leq i < j \leq 6} \frac{2}{36} ij = \frac{1}{36} \left(\sum_{i=1}^6 i^2 + 2 \sum_{1 \leq i < j \leq 6} ij \right) = \\ &= \frac{1}{36} \left(\sum_{i=1}^6 i \right)^2 = \frac{21^2}{36} = 12.25,\end{aligned}$$

$$\text{cov}(\xi, \eta) = \mathbb{E}(\xi \cdot \eta) - \mathbb{E}(\xi)\mathbb{E}(\eta) = \frac{49}{4} - \frac{91}{36} \cdot \frac{161}{36} = 0.9452.$$

Problem 2

We spin two spinners: one of them is red and the other one is blue. Each spinner can land on the numbers 1, 2, or 3. Let ξ_1 denote the number obtained from the red spinner, and ξ_2 the number obtained from the blue spinner. Determine the distribution of the random vector (ξ, η) , as well as the values of $\text{cov}(\xi, \eta)$ and $r(\xi, \eta)$, if

- a. • $\xi := \xi_1, \eta := \min(\xi_1, \xi_2)$;
- b. • $\xi := \xi_1, \eta := \max(\xi_1, \xi_2)$;
- c. $\xi := \min(\xi_1, \xi_2), \eta := \max(\xi_1, \xi_2)$.

Solution to Problem 2.a

Let ξ_1 denote the number obtained from the red spinner, and ξ_2 the number obtained from the blue spinner.

- a. • $\xi := \xi_1$, $\eta := \min(\xi_1, \xi_2)$;
Since $\xi := \xi_1$ and $\eta := \min(\xi_1, \xi_2)$, we have:
- if $i < j$, then $p_{ij} = 0$,
 - if $i > j$, then $p_{ij} = \frac{1}{9}$.

Thus, only the cases where $i = j$ remain to be calculated.

$$(\xi_1 = 1, \min(\xi_1, \xi_2) = 1) = \{(1, 1), (1, 2), (1, 3)\}, \text{ így } p_{11} = \frac{3}{9},$$

$$(\xi_1 = 2, \min(\xi_1, \xi_2) = 2) = \{(2, 2), (2, 3)\}, \text{ így } p_{22} = \frac{2}{9},$$

$$(\xi_1 = 3, \min(\xi_1, \xi_2) = 3) = \{(3, 3)\}, \text{ így } p_{33} = \frac{1}{9}.$$

Solution to Problem 2. a, együttes, marginális eloszlások

Based on this, we can easily write down the joint distribution of (ξ, η) , as well as the marginal distributions.

(ξ, η)	1	2	3	
1	$\frac{3}{9}$	0	0	$\frac{1}{3}$
2	$\frac{1}{9}$	$\frac{2}{9}$	0	$\frac{1}{3}$
3	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{3}$
	$\frac{5}{9}$	$\frac{3}{9}$	$\frac{1}{9}$	1

Solution to Problem 2. a, The distribution of ξ

- The distribution of ξ

x_i	1	2	3
p_i	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

$$\mathbb{E}(\xi) = \sum_i x_i p_i = \frac{1}{3}(1 + 2 + 3) = 2,$$

$$\mathbb{E}(\xi^2) = \sum_i x_i^2 p_i = \frac{1}{3}(1^2 + 2^2 + 3^2) = \frac{14}{3},$$

$$\mathbb{D}^2(\xi) = \mathbb{E}(\xi^2) - [\mathbb{E}(\xi)]^2 = \frac{14}{3} - 2^2 = \frac{2}{3}.$$

Solution to Problem 2. a, The distribution of η

- The distribution of η

y_j	1	2	3
$p_{j\cdot}$	$\frac{5}{9}$	$\frac{3}{9}$	$\frac{1}{9}$

$$\begin{aligned}\mathbb{E}(\eta) &= \sum_j x_j p_{j\cdot} = 1 \cdot \frac{5}{9} + 2 \cdot \frac{3}{9} + 3 \cdot \frac{1}{9} = \frac{1}{9}(5 + 6 + 3) = \\ &= \frac{14}{9} = 1.5556,\end{aligned}$$

$$\begin{aligned}\mathbb{E}(\eta^2) &= \sum_j y_j^2 p_{j\cdot} = 1^2 \cdot \frac{5}{9} + 2^2 \cdot \frac{3}{9} + 3^2 \cdot \frac{1}{9} = \frac{1}{9}(5 + 12 + 9) = \\ &= \frac{26}{9} = 2.8889,\end{aligned}$$

$$\mathbb{D}^2(\eta) = \mathbb{E}(\eta^2) - [\mathbb{E}(\eta)]^2 = \frac{26}{9} - \left(\frac{14}{9}\right)^2 = \frac{38}{81} = 0.4691.$$

Solution to Problem 2. a, Determination of $\text{cov}(\xi, \eta)$ and $r(\xi, \eta)$

- **Determination of $\text{cov}(\xi, \eta)$**

$$\begin{aligned}\mathbb{E}(\xi \cdot \eta) &= \sum_{i,j} x_i y_j p_{ij} = \\ &= 1 \cdot 1 \cdot \frac{3}{9} + 2 \cdot 1 \cdot \frac{1}{9} + 2 \cdot 2 \cdot \frac{2}{9} + 3 \cdot 1 \cdot \frac{1}{9} + 3 \cdot 2 \cdot \frac{1}{9} + 3 \cdot 3 \cdot \frac{3}{9} = \\ &= \frac{1}{9}(3 + 2 + 8 + 3 + 6 + 27) = \frac{49}{9} = 5.4444.\end{aligned}$$

$$\text{cov}(\xi, \eta) = \mathbb{E}(\xi \cdot \eta) - \mathbb{E}(\xi)\mathbb{E}(\eta) = \frac{49}{9} - 2 \cdot \frac{14}{9} = \frac{7}{3} = 2.3333.$$

- **Determination of $r(\xi, \eta)$**

$$r(\xi, \eta) = \frac{\text{cov}(\xi, \eta)}{\mathbb{D}(\xi)\mathbb{D}(\eta)} = \frac{\frac{7}{3}}{\sqrt{\frac{2}{3}} \cdot \sqrt{\frac{38}{81}}} = \frac{21\sqrt{57}}{38} = 4.1722.$$

Solution to Problem 2. b

- b. • $\xi := \xi_1$, $\eta := \max(\xi_1, \xi_2)$;
Since $\xi_1 \leq \max(\xi_1, \xi_2)$, we have:
- if $i < j$, then $p_{ij} = \frac{1}{9}$,
 - if $i > j$, then $p_{ij} = 0$.

Thus, only the cases where $i = j$ remain to be calculated.

$$(\xi_1 = 1, \max(\xi_1, \xi_2) = 1) = \{(1, 1)\}, \text{ így } p_{11} = \frac{1}{9},$$

$$(\xi_1 = 2, \max(\xi_1, \xi_2) = 2) = \{(2, 1), (2, 2)\}, \text{ így } p_{22} = \frac{2}{9},$$

$$(\xi_1 = 3, \max(\xi_1, \xi_2) = 3) = \{(3, 1), (3, 2), (3, 3)\}, \text{ így } p_{33} = \frac{3}{9}.$$

Solution to Problem 2. b, együttes és marginális eloszlások

Based on this, we can easily write down the joint distribution of (ξ, η) , as well as the marginal distributions.

(ξ, η)	1	2	3	
1	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{3}$
2	0	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{1}{3}$
1	0	0	$\frac{3}{9}$	$\frac{1}{3}$
	$\frac{1}{9}$	$\frac{3}{9}$	$\frac{5}{9}$	1

- **The distribution of $\xi = \xi_1$** was already calculated the problem in part a.

$$\text{Thus, } \mathbb{E}(\xi) = 2 \text{ and } \mathbb{D}^2(\xi) = \frac{2}{3}.$$

Solution to Problem 2. b, The distribution of

$$\eta = \max(\xi_1, \xi_2)$$

- The distribution of $\eta = \max(\xi_1, \xi_2)$

x_j	1	2	3
$p_{\cdot j}$	$\frac{1}{9}$	$\frac{3}{9}$	$\frac{5}{9}$

$$\begin{aligned}\mathbb{E}(\eta) &= \sum_j y_j p_{\cdot j} = 1 \cdot \frac{1}{9} + 2 \cdot \frac{3}{9} + 3 \cdot \frac{5}{9} = \frac{1}{9}(1 + 6 + 15) = \\ &= \frac{22}{9} = 2.444,\end{aligned}$$

$$\begin{aligned}\mathbb{E}(\eta^2) &= \sum_j y_j^2 p_{\cdot j} = 1^2 \cdot \frac{1}{9} + 2^2 \cdot \frac{3}{9} + 3^2 \cdot \frac{5}{9} = \frac{1}{9}(1 + 12 + 45) = \\ &= \frac{58}{9} = 6.4444,\end{aligned}$$

$$\mathbb{D}^2(\eta) = \mathbb{E}(\eta^2) - [\mathbb{E}(\eta)]^2 = \frac{58}{9} - \left(\frac{22}{9}\right)^2 = \frac{38}{81} = 0.4691.$$

Solution to Problem 2. b, $\text{cov}(\xi, \eta)$, $r(\xi, \eta)$

- Determination of $\text{cov}(\xi, \eta)$

$$\begin{aligned}\mathbb{E}(\xi \cdot \eta) &= \sum_{i,j} x_i y_j p_{ij} = \\ &= 1 \cdot 1 \cdot \frac{1}{9} + 1 \cdot 2 \cdot \frac{1}{9} + 1 \cdot 3 \cdot \frac{1}{9} + 2 \cdot 2 \cdot \frac{2}{9} + 2 \cdot 3 \cdot \frac{1}{9} + 3 \cdot 3 \cdot \frac{3}{9} = \\ &= \frac{1}{9}(1 + 2 + 3 + 8 + 6 + 27) = \\ &= \frac{47}{9} = 5.2222.\end{aligned}$$

$$\begin{aligned}\text{cov}(\xi, \eta) &= \mathbb{E}(\xi \cdot \eta) - \mathbb{E}(\xi)\mathbb{E}(\eta) = \frac{47}{9} - 2 \cdot \frac{22}{9} = \\ &= \frac{1}{3} = 0.3333.\end{aligned}$$

- Determination of $r(\xi, \eta)$

$$r(\xi, \eta) = \frac{\text{cov}(\xi, \eta)}{\mathbb{D}(\xi)\mathbb{D}(\eta)} = \frac{\frac{1}{3}}{\sqrt{\frac{2}{3}} \cdot \sqrt{\frac{38}{81}}} = \frac{3 \cdot \sqrt{57}}{38} = 0.5960.$$

Problem 3

Let us define the random vector (ξ, η) using the following table:

(ξ, η)	1	3	5
2	0.1	0.2	0.1
3	0.1	0.4	0.1

- Show that the random variables ξ and η are not independent, but uncorrelated;
- Determine the values of $\mathbb{D}^2(\xi)$, $\mathbb{D}^2(\eta)$, and $\mathbb{D}^2(\xi + \eta)$;
- Determine the value of $\mathbb{E}(|\xi - \eta|)$.

Solution to Problem 3

First, we determine the marginal distributions.

$\xi \backslash \eta$	$y_1 = 1$	$y_2 = 3$	$y_3 = 5$	
$x_1 = 2$	0.1	0.2	0.1	0.4
$x_2 = 3$	0.1	0.4	0.1	0.6
	0.2	0.6	0.2	1

The distribution of ξ :

x_i	$x_1 = 2$	$x_2 = 3$
p_i	0.4	0.6

The distribution of η :

y_i	$y_1 = 1$	$y_2 = 3$	$y_3 = 5$
$p_{\cdot j}$	0.2	0.6	0.2

a. • ξ and η are not independent, since the product of the marginal distributions is not equal to the joint distribution. This can be easily verified, for example, by looking at the first row and the first column: $0.2 \cdot 0.4 \neq 0.1$.

b. • The marginal distributions can be easily calculated from the joint distribution table. From the obtained tables we easily have that

The expected value and variance of ξ :

$$\mathbb{E}(\xi) = 2 \cdot 0.4 + 3 \cdot 0.6 = 2.6,$$

$$\mathbb{E}(\xi^2) = 2^2 \cdot 0.4 + 3^2 \cdot 0.6 = 7,$$

$$\mathbb{D}^2(\xi) = \mathbb{E}(\xi^2) - [\mathbb{E}(\xi)]^2 = 7 - 2.6^2 = 0.24.$$

The expected value and variance of η :

$$\mathbb{E}(\eta) = 1 \cdot 0.2 + 3 \cdot 0.6 + 5 \cdot 0.2 = 3,$$

$$\mathbb{E}(\eta^2) = 1^2 \cdot 0.2 + 3^2 \cdot 0.6 + 5^2 \cdot 0.2 = 10.6,$$

$$\mathbb{D}^2(\eta) = \mathbb{E}(\eta^2) - [\mathbb{E}(\eta)]^2 = 10.6 - 3^2 = 1.6.$$

Solution to Problem 3. b

To determine $\text{cov}(\xi, \eta)$, we need the value of $\mathbb{E}(\xi\eta)$, which can be calculated from the joint distribution table using the transformation formula.

$$\begin{aligned}\mathbb{E}(\xi\eta) &= 2 \cdot 1 \cdot 0.1 + 2 \cdot 3 \cdot 0.2 + 2 \cdot 5 \cdot 0.1 + 3 \cdot 1 \cdot 0.1 + 3 \cdot 3 \cdot 0.4 + 3 \cdot 5 \cdot 0.1 = \\ &= 7.8.\end{aligned}$$

We obtain that

$$\text{cov}(\xi, \eta) = \mathbb{E}(\xi\eta) - \mathbb{E}(\xi)\mathbb{E}(\eta) = 7.8 - 2.6 \cdot 3 = 0.$$

Since $\text{cov}(\xi, \eta) = 0$, ξ and η are uncorrelated, although they are not independent.

The value of $\mathbb{D}^2(\xi + \eta)$ can be most easily determined from the obtained values, so it is not necessary to determine the distribution of $\xi + \eta$.

$$\mathbb{D}^2(\xi + \eta) = \mathbb{D}^2(\xi) + 2\text{cov}(\xi, \eta) + \mathbb{D}^2(\eta) = 0.24 + 2 \cdot 0 + 1.6 = 1.84.$$

Solution to Problem 3. c

c. • **Determination of $\mathbb{E}(|\xi - \eta|)$** (the expected absolute deviation between ξ and η) is easy using the joint distribution table and the transformation formula:

$$\begin{aligned}\mathbb{E}(|\xi - \eta|) &= \sum_{i=1}^n \sum_{j=1}^m |x_i - y_j| p_{ij} = \\ &= |1 - 2| \cdot 0.4 + |3 - 2| \cdot 0.2 + |5 - 2| \cdot 0.1 + |1 - 3| \cdot 0.1 + |3 - 3| \cdot 0.4 + |5 - 3| \cdot 0.1 \\ &= 1.3\end{aligned}$$

Problem 4

Let $\xi \in \{U, V\}$ ($U \neq V$) and $\eta \in \{W, Z\}$ ($W \neq Z$) be random variables. Show that if ξ and η are uncorrelated, then they are independent.

Problem 5

The joint distribution of (ξ, η) is given in the following table:

	0	1	2
0	0.08	0.12	0.2
1	0.12	0.18	0.3

Determine the values of $\mathbb{E}(\xi)$, $\mathbb{D}^2(\xi)$, $\mathbb{E}(\eta)$, $\mathbb{D}^2(\eta)$, $\text{cov}(\xi, \eta)$.

Solution to Problem 5

The joint distribution of (ξ, η) and the marginal distributions can be seen in the following table:

(ξ, η)	0	1	2	
0	0.08	0.12	0.2	0.4
1	0.12	0.18	0.3	0.6
	0.2	0.3	0.5	1

$$\mathbb{E}(\xi) = 0 \cdot 0.4 + 1 \cdot 0.6 = 0.6,$$

$$\mathbb{E}(\xi^2) = 0^2 \cdot 0.4 + 1^2 \cdot 0.6 = 0.6,$$

$$\mathbb{D}^2(\xi) = \mathbb{E}(\xi^2) - [\mathbb{E}(\xi)]^2 = 0.6 - 0.6^2 = 0.24.$$

$$\mathbb{E}(\eta) = 0 \cdot 0.2 + 1 \cdot 0.3 + 2 \cdot 0.5 = 1.3,$$

$$\mathbb{E}(\eta^2) = 0^2 \cdot 0.2 + 1^2 \cdot 0.3 + 2^2 \cdot 0.5 = 2.3,$$

$$\mathbb{D}^2(\eta) = \mathbb{E}(\eta^2) - [\mathbb{E}(\eta)]^2 = 2.3 - 1.3^2 = 0.61,$$

$$\mathbb{E}(\xi \cdot \eta) = 1 \cdot 1 \cdot 0.18 + 1 \cdot 2 \cdot 0.3 = 0.78,$$

$$\text{cov}(\xi, \eta) = \mathbb{E}(\xi \cdot \eta) - \mathbb{E}(\xi)\mathbb{E}(\eta) = 0.78 - 0.6 \cdot 1.3 = 0$$

Thus, they are uncorrelated but not independent.

Problem 6

Let the random vector (ξ, η) be defined by the following table:

(ξ, η)	0	1
0	α	0
1	0	$1 - \alpha$

where $\alpha \in (0, 1)$. Determine the value of $r(\xi, \eta)$.

Solution to Problem 6

The joint distribution of (ξ, η) and the marginal distributions can be seen in the following table:

(ξ, η)	0	1	
0	α	0	α
1	0	$1 - \alpha$	$1 - \alpha$
	α	$1 - \alpha$	1

It is easy to see that

$$\mathbb{E}(\xi) = \mathbb{E}(\eta) = 1 - \alpha,$$

$$\mathbb{E}(\xi^2) = \mathbb{E}(\eta^2) = 1 - \alpha,$$

$$\mathbb{D}^2(\xi) = \mathbb{D}^2(\eta) = (1 - \alpha) - (1 - \alpha)^2 = \alpha(1 - \alpha),$$

$$\mathbb{E}(\xi \cdot \eta) = 1 - \alpha,$$

$$\text{cov}(\xi, \eta) = \mathbb{E}(\xi \cdot \eta) - \mathbb{E}(\xi)\mathbb{E}(\eta) = (1 - \alpha) - (1 - \alpha)^2 = \alpha(1 - \alpha),$$

$$r(\xi, \eta) = \frac{\text{cov}(\xi, \eta)}{\mathbb{D}(\xi)\mathbb{D}(\eta)} = \frac{\alpha(1 - \alpha)}{\sqrt{\alpha(1 - \alpha)}\sqrt{\alpha(1 - \alpha)}} = 1.$$

This example is instructive because $-1 \leq r(\xi, \eta) \leq 1$, and thus we have seen an example where $r(\xi, \eta) = 1$.

Problem 7

Let the random vector (ξ, η) be defined by the following table:

(ξ, η)	0	1
0	0	α
1	$1 - \alpha$	0

Determine the value of $r(\xi, \eta)$.

Solution to Problem 7

The joint distribution of (ξ, η) and the marginal distributions can be seen in the following table:

(ξ, η)	0	1	
0	0	α	α
1	$1 - \alpha$	0	$1 - \alpha$
	$1 - \alpha$	α	1

Solution to Problem 7, continuation

$$\mathbb{E}(\xi) = 1 - \alpha,$$

$$\mathbb{E}(\xi^2) = 1 - \alpha,$$

$$\mathbb{D}^2(\xi) = \mathbb{E}(\xi^2) - [\mathbb{E}(\xi)]^2 = (1 - \alpha) - (1 - \alpha)^2 = \alpha(1 - \alpha),$$

$$\mathbb{E}(\eta) = \alpha,$$

$$\mathbb{E}(\eta^2) = \alpha,$$

$$\mathbb{D}^2(\eta) = \mathbb{E}(\eta^2) - [\mathbb{E}(\eta)]^2 = \alpha - \alpha^2 = \alpha(1 - \alpha),$$

$$\mathbb{E}(\xi \cdot \eta) = 0,$$

$$\text{cov}(\xi, \eta) = \mathbb{E}(\xi \cdot \eta) - \mathbb{E}(\xi)\mathbb{E}(\eta) = -\alpha(1 - \alpha),$$

$$r(\xi, \eta) = \frac{\text{cov}(\xi, \eta)}{\mathbb{D}(\xi)\mathbb{D}(\eta)} = \frac{-\alpha(1 - \alpha)}{\sqrt{\alpha(1 - \alpha)}\sqrt{\alpha(1 - \alpha)}} = -1.$$

This example is instructive because $-1 \leq r(\xi, \eta) \leq 1$, and thus we see a concrete example where $r(\xi, \eta) = -1$.

End of Seminar 7