

# Probability Theory and Mathematical Statistics

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## Seminar 6

# Famous absolutely continuous random variables

# 1. Uniform distribution

# Problem 1

Compute the distribution, and the density function, expected value, and variance of the random variable  $\eta = \xi^2$  if

- a.  $\xi \sim \mathcal{U}(0, 1)$ ;
- b.  $\xi \sim \mathcal{U}(-1, 1)$ .

## Solution to Problem 1

Let  $\eta = \xi^2$ . Investigate the cases when

- a.  $\xi \sim \mathcal{U}(0, 1)$ . Let  $x \in [0, 1]$ . Then

$$\mathbb{F}_\eta(x) = \mathbb{P}(\eta < x) = \mathbb{P}(\xi^2 < x) = \mathbb{P}(\xi < \sqrt{x}) = \mathbb{F}_\xi(\sqrt{x})$$

- The distribution and density function:

$$\mathbb{F}_\eta(x) = \begin{cases} 0, & \text{if } x < 0, \\ \frac{\sqrt{x} - 0}{1 - 0} = \sqrt{x}, & \text{if } x \in [0, 1], \\ 1, & \text{if } x > 1. \end{cases}$$

Since  $f(x) = F'(x)$ , we obtain that

$$f_\eta(x) = \begin{cases} \frac{1}{2\sqrt{x}}, & \text{if } x \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

## Problem 1.a, continuation

- the expected value, and the second moment of  $\eta$ :

$$\begin{aligned}\mathbb{E}(\eta) &= \int_{-\infty}^{\infty} xf(x)dx = \frac{1}{2} \int_0^1 x \frac{1}{\sqrt{x}} dx = \frac{1}{2} \int_0^1 x^{\frac{1}{2}} dx = \\ &= \frac{1}{2} \cdot \left[ \frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_{x=0}^{x=1} = \frac{1}{3} = 0.3333,\end{aligned}$$

$$\begin{aligned}\mathbb{E}(\eta^2) &= \int_{-\infty}^{\infty} x^2 f(x)dx = \frac{1}{2} \int_0^1 x^2 \frac{1}{\sqrt{x}} dx = \frac{1}{2} \int_0^1 x^{\frac{3}{2}} dx = \\ &= \frac{1}{2} \cdot \left[ \frac{x^{\frac{5}{2}}}{\frac{5}{2}} \right]_{x=0}^{x=1} = \frac{1}{5} = 0.2.\end{aligned}$$

- The variance of  $\eta$ :

$$\mathbb{D}^2(\eta) = \mathbb{E}(\eta^2) - (\mathbb{E}(\eta))^2 = \frac{1}{5} - \left(\frac{1}{3}\right)^2 = \frac{4}{45} = 0.0889.$$

## Solution to Problem 1.b

b.  $\xi \sim \mathcal{U}(-1, 1)$ .

Let  $x \in ]-\infty, 0[$ . Then

$$\mathbb{F}_\eta(x) = \mathbb{P}(\eta < x) = \mathbb{P}(\xi^2 < x) = 0.$$

Let  $x \in [0, 1]$ . Then

$$\begin{aligned}\mathbb{F}_\eta(x) &= \mathbb{P}(\eta < x) = \mathbb{P}(\xi^2 < x) = \mathbb{P}(|\xi| < \sqrt{x}) = \\ &= \mathbb{P}(-\sqrt{x} < \xi < \sqrt{x}) = \mathbb{F}_\xi(\sqrt{x}) - \mathbb{F}_\xi(-\sqrt{x}) = \\ &= \frac{\sqrt{x} + 1}{2} - \frac{-\sqrt{x} + 1}{2} = \sqrt{x}.\end{aligned}$$

## Problem 1.b, continuation

- The distribution and density function:

$$\mathbb{F}_\eta(x) = \begin{cases} 0, & \text{if } x < 0, \\ \sqrt{x}, & \text{if } x \in [0, 1], \\ 1, & \text{if } x > 1. \end{cases}$$

Thus

$$f_\eta(x) = \begin{cases} \frac{1}{2\sqrt{x}}, & \text{if } x \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

- Expected value and variance are the same as in the previous problem:  $\mathbb{E}(\eta) = \frac{1}{3}$ ,  $\mathbb{D}^2(\eta) = \frac{4}{45}$ .

## Problem 2

Let  $\xi \sim \mathcal{U}(a, b)$ ;  $\alpha > 0$ ;  $\beta \in \mathbb{R}$ .

- a. Determine the distribution of the random variable defined by  $\eta := \alpha\xi + \beta$ ;
- b. How to choose  $\alpha$  and  $\beta$  so that  $\eta \sim \mathcal{U}(0, 1)$ .

## Solution to Problem 2.a

Let  $\eta := \alpha\xi + \beta$  where  $\xi \sim \mathcal{U}(a, b)$ ;  $\alpha > 0$ ;  $\beta \in \mathbb{R}$ .

a. The distribution of  $\eta$ :

$$\mathbb{F}_\eta(x) = \mathbb{P}(\eta < x) = \mathbb{P}(\alpha\xi + \beta < x) = \mathbb{P}\left(\xi < \frac{x - \beta}{\alpha}\right) = \mathbb{F}_\xi\left(\frac{x - \beta}{\alpha}\right) \quad (x \in \mathbb{R}).$$

Thus

$$\mathbb{F}_\eta(x) = \begin{cases} 0, & \text{if } x < \alpha a + \beta, \\ \frac{\frac{x - \beta}{\alpha} - a}{b - a} = \frac{x - (\alpha a + \beta)}{(\alpha b + \beta) - (\alpha a + \beta)}, & \text{if } x \in [\alpha a + \beta, \alpha b + \beta], \\ 1, & \text{if } x > \alpha b + \beta. \end{cases}$$

Hence  $\eta \sim \mathcal{U}(\alpha a + \beta, \alpha b + \beta)$ .

## Solution to Problem 2.b

b. We have to solve the system

$$\left. \begin{aligned} \alpha a + \beta &= 0 \\ \alpha b + \beta &= 1 \end{aligned} \right\}$$

Subtract the first equation from the second one we obtain that  $\alpha(b - a) = 1$ , thus we obtain that  $\alpha = \frac{1}{b-a}$ .

Substitute the obtained value of  $\alpha$  back into the first equation, then we obtain that

$$\frac{1}{b-a}a + \beta = 0 \quad \implies \quad \beta = \frac{-a}{b-a}.$$

## 2. Exponential distribution

## Problem 3

The probability that a customer waits more than 5 minutes at a petrol station is 0.22. Assuming that the waiting time is an exponential distributed random variable, what is the probability that a customer waits less than 6 minutes?

## Solution of Problem 3

Let  $\eta$  denote the waiting time measured in minutes. Since  $\eta \sim \text{Exp}(\lambda)$ , we have that

$$F_{\eta}(x) = 1 - e^{-\lambda x}, \quad (x > 0).$$

**Determination of the parameter  $\lambda$ :**

$$0.22 = \mathbb{P}(\eta > 5) = 1 - \mathbb{P}(\eta < 5) = 1 - F_{\eta}(5) = 1 - (1 - e^{-\lambda \cdot 5}) = e^{-5\lambda},$$

from which we get that  $-5\lambda = \ln(0.22)$ , that is

$$\lambda = \frac{\ln(0.22)}{-5}.$$

**The desired probability:**

$$\mathbb{P}(\eta < 6) = F_{\eta}(6) = 1 - e^{-\lambda \cdot 6} = 1 - e^{\frac{6 \cdot \ln(0.22)}{5}} = 0.8375.$$

## Problem 4

A machine contains  $n$  components, whose failure causes the machine to stop. The lifetimes of the individual components are independent exponentially distributed random variables with parameters  $\lambda_1, \lambda_2, \dots, \lambda_n$ .

Let  $\eta$  denote the time interval that elapses from starting the machine until the first shutdown.

Calculate the density function and the expected value of  $\eta$ .

## Solution of Problem 4

Let  $\xi_1, \xi_2, \dots, \xi_n$  be independent random variables with distribution functions  $F_1, F_2, \dots, F_n$ . If  $\xi := \min(\xi_1, \xi_2, \dots, \xi_n)$ , then the distribution function of the random variable  $\xi$  is:

$$\begin{aligned}\mathbb{P}(\xi < x) &= \mathbb{P}(\min(\xi_1, \xi_2, \dots, \xi_n) < x) \\ &= 1 - \mathbb{P}(\min(\xi_1, \xi_2, \dots, \xi_n) > x) \\ &= 1 - \mathbb{P}(\xi_1 > x, \xi_2 > x, \dots, \xi_n > x) \\ &= 1 - \mathbb{P}(\xi_1 > x)\mathbb{P}(\xi_2 > x) \dots \mathbb{P}(\xi_n > x) \\ &= 1 - (1 - \mathbb{P}(\xi_1 < x))(1 - \mathbb{P}(\xi_2 < x)) \dots (1 - \mathbb{P}(\xi_n < x)).\end{aligned}$$

Equivalently we obtain that

$$\mathbb{F}(x) = 1 - (1 - \mathbb{F}_1(x))(1 - \mathbb{F}_2(x)) \dots (1 - \mathbb{F}_n(x)).$$

## Solution of Problem 4, Continuation

In the special case which is investigated by us we obtain that

$$\begin{aligned} F(x) &= 1 - (1 - F_1(x))(1 - F_2(x)) \dots (1 - F_n(x)) \\ &= 1 - e^{-\lambda_1 x} e^{-\lambda_2 x} \dots e^{-\lambda_n x} \\ &= 1 - e^{-(\lambda_1 + \lambda_2 + \dots + \lambda_n)x}. \end{aligned}$$

Hence, the minimum of independent exponentially distributed random variables is also an exponentially distributed random variable with parameter  $\lambda = \lambda_1 + \lambda_2 + \dots + \lambda_n$ . Therefore, the expected value of the lifetime of the machine is

$$\mathbb{E}(\xi) = \frac{1}{\lambda_1 + \lambda_2 + \dots + \lambda_n}.$$

## Problem 5

Let  $\xi$  be a random variable with exponential distribution such that the standard deviation of  $\xi$  is 0.30. Determine the value of

$$\mathbb{E}(8\xi^2 - 19\xi + 7).$$

## Solution of Problem 5

$\xi \sim \text{Exp}(\lambda)$ . We know that  $\mathbb{E}(\xi) = \mathbb{D}(\xi) = 0.3 = \frac{1}{\lambda}$ . Thus,

$$\frac{1}{\lambda^2} = \mathbb{D}^2(\xi) = \mathbb{E}(\xi^2) - [\mathbb{E}(\xi)]^2 = \mathbb{E}(\xi^2) - \frac{1}{\lambda^2},$$

from which we get that

$$\mathbb{E}(\xi^2) = \frac{2}{\lambda^2}.$$

By linearity of expectation:

$$\begin{aligned}\mathbb{E}(8\xi^2 - 19\xi + 7) &= 8\mathbb{E}(\xi^2) - 19\mathbb{E}(\xi) + 7 = 8 \cdot \frac{2}{\lambda^2} - 19 \cdot \frac{1}{\lambda} + 7 = \\ &= 16 \cdot 0.3^2 - 19 \cdot 0.3 + 7 = 2.74.\end{aligned}$$

## Problem 6

In the driving exam, the exam duration (measured in minutes) is a random variable with density function

$$f(x) = \begin{cases} 0, & \text{if } x < 0, \\ 0.02 \cdot e^{-0.02x}, & \text{if } x > 0. \end{cases}$$

The person before us has already been driving for 13 minutes. What is the probability that they will not finish the exam within the next 6 minutes?

## Solution of Problem 6

Let  $\xi$  denote the exam duration measured in minutes, then

$$\xi \sim \text{Exp}(\lambda = 0.02).$$

In the calculation, we use the memoryless property of the exponential distribution, which states:

$$\mathbb{P}(\xi < s + t \mid \xi > s) = \mathbb{P}(\xi < t), \quad (s, t > 0).$$

The desired probability is

$$\begin{aligned} \mathbb{P}(\xi > 6 + 13 \mid \xi > 13) &= 1 - \mathbb{P}(\xi < 6 + 13 \mid \xi > 13) \\ &= 1 - \mathbb{P}(\xi < 6) \\ &= 1 - F_{\xi}(6) \\ &= 1 - (1 - e^{-0.02 \cdot 6}) \\ &= e^{-0.02 \cdot 6} = 0.8869. \end{aligned}$$

## Problem 7

On average, 29 customers arrive at a shop in an hour, and their number is Poisson distributed. What is the probability that at least 4.5 minutes elapse between two consecutive customers?

## Solution of Problem 7

The time unit is 1 hour. Let  $\xi$  denote the number of customers arriving at the shop per hour.

Since  $\mathbb{E}(\xi) = \lambda = 29$ , we have that  $\xi \sim \text{Poiss}(\lambda = 29)$ .

Let  $\eta$  denote the time elapsed between two consecutive customers measured in hours. Then by the Poisson process we have that  $\eta \sim \text{Exp}(\lambda = 29)$ . The desired probability is

$$\begin{aligned}\mathbb{P}\left(\eta > \frac{4.5}{60}\right) &= 1 - \mathbb{P}\left(\eta < \frac{4.5}{60}\right) = 1 - F_{\eta}\left(\frac{4.5}{60}\right) = \\ &= 1 - \left(1 - e^{-\frac{4.5 \cdot 29}{60}}\right) = e^{-\frac{4.5 \cdot 29}{60}} = 0.1136.\end{aligned}$$

### 3. Normal Distribution

## Problem 8

The length of a workpiece is approximately a normally distributed random variable with expected value 50 mm.

Determine the standard deviation of the length of a workpiece if the probability that the length of the workpiece is less than 50.05 mm is 0.85.

## Solution of Problem 8

Let  $\xi$  denote the length of the workpiece. We know that  $\xi \sim \mathcal{N}(m = 50, \sigma^2)$ .

First we have to determine the value of the unknown standard deviation. We have to apply the "standardization".

$$0.85 = \mathbb{P}(\xi < 50.05) = \mathbb{P}\left(\eta < \frac{50.05 - 50}{\sigma}\right) = \Phi\left(\frac{0.05}{\sigma}\right),$$

from which the unknown parameter  $\sigma$  can be easily expressed:

$$1.04 = \frac{0.05}{\sigma}, \quad \sigma = \frac{0.05}{1.04} = 0.0481.$$

## Problem 9

Let  $\xi$  be a normally distributed random variable with zero expected value,  $\xi$  takes values in the interval  $(-5.8, 5.8)$  with probability of 0.42. Calculate the probability

$$\mathbb{P}(0.5 \leq 1.7\xi < 1.7).$$

## Solution of Problem 9

We know that  $\xi \sim \mathcal{N}(m = 0, \sigma^2)$ , where  $\sigma^2$  is unknown; furthermore,

$$\mathbb{P}(-5.8 \leq \xi \leq 5.8) = 0.42.$$

First, we determine the value of  $\sigma$ . With standardizing we get that

$$\eta := \frac{\xi - 0}{\sigma} \sim \mathcal{N}(0, 1).$$

$$\begin{aligned} \mathbb{P}(-5.8 \leq \xi \leq 5.8) &= \mathbb{P}\left(-\frac{5.8}{\sigma} \leq \eta \leq \frac{5.8}{\sigma}\right) = \\ &= \Phi\left(\frac{5.8}{\sigma}\right) - \Phi\left(-\frac{5.8}{\sigma}\right) = 2\Phi\left(\frac{5.8}{\sigma}\right) - 1 = 0.42, \end{aligned}$$

which gives that

$$\Phi\left(\frac{5.8}{\sigma}\right) = \frac{0.42 + 1}{2} = 0.71.$$

## Solution of Problem 9, Continuation

From the obtained equality, the unknown standard deviation can be easily expressed

$$\frac{5.8}{\sigma} = 0.55 \quad \Rightarrow \quad \sigma = \frac{5.8}{0.55} = 10.5455.$$

The desired probability can also be easily expressed by "standardization"

$$\begin{aligned} \mathbb{P}(0.5 \leq 1.7\xi + 1 < 1.7) &= \mathbb{P}\left(\frac{0.5 - 1}{1.7} \leq \xi < \frac{1.7 - 1}{1.7}\right) \\ &= \mathbb{P}\left(\frac{0.5 - 1}{1.7 \cdot 10.5455} \leq \eta < \frac{1.7 - 1}{1.7 \cdot 10.5455}\right) = \Phi(0.04) - \Phi(-0.03) \\ &= \Phi(0.04) + \Phi(0.03) - 1 = 0.5160 + 0.5160 - 1 = 0.028. \end{aligned}$$

## Problem 10

Let  $\xi \sim \mathcal{N}(m, \sigma^2)$  be a normally distributed random variable, and let  $\alpha \in (0, 1)$  be an arbitrarily fixed number. Determine the number  $K_\alpha$  such that

$$\mathbb{P}(\xi > K_\alpha) = \alpha.$$

Specifically, let  $m = 20$  and  $\sigma^2 = 4$ . Find the values of  $K_\alpha$  for  $\alpha = 0.90, 0.95, 0.99$ .

## Solution of Problem 10

We have that

$$\begin{aligned}\mathbb{P}(\xi > K_\alpha) &= 1 - \mathbb{P}(\xi < K_\alpha) = 1 - \mathbb{P}\left(\frac{\xi - m}{\sigma} < \frac{K_\alpha - m}{\sigma}\right) = \\ &= 1 - \Phi\left(\frac{K_\alpha - m}{\sigma}\right) = \Phi\left(-\frac{K_\alpha - m}{\sigma}\right) = \alpha\end{aligned}$$

which implies that

$$-\frac{K_\alpha - m}{\sigma} = \Phi^{-1}(\alpha) \quad \Rightarrow \quad K_\alpha = -\sigma\Phi^{-1}(\alpha) + m.$$

The following table shows the values of  $\alpha$  and  $\Phi^{-1}(\alpha)$ :

$\alpha$	0.90	0.95	0.99
$\Phi^{-1}(\alpha)$	1.282	0.645	2.326

## Solution of Problem 10, continuation

Given  $m = 20$  and  $\sigma = 2$ :

$$\text{If } \alpha = 0.90 \quad \text{then} \quad K_\alpha = -2 \cdot 1.282 + 20 = 17.436,$$

$$\text{If } \alpha = 0.95 \quad \text{then} \quad K_\alpha = -2 \cdot 1.645 + 20 = 16.71,$$

$$\text{If } \alpha = 0.99 \quad \text{then} \quad K_\alpha = -2 \cdot 2.326 + 20 = 15.348.$$

End of Seminar 6