Probability Theory and Mathematical Statistics

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Seminar 5

Famous Discrete Random Variables

1. Hypergeometric and Binomial Distributions

Problem 1

The probability of occurrence of an event A is 0.31. What is the probability that in 8 trials

a. it occurs at most 2 times,

b. it occurs at most 6 times,

c. it occurs at least 6 times.

Solution to Problem 1.a,b

The probability of occurrence of an event A is 0.31. Let ξ denote the number of occurrences of event A.

a.

$$\begin{aligned}\mathbb{P}(\xi \leqslant 2) &= \mathbb{P}((\xi = 0) \cup (\xi = 1) \cup (\xi = 2)) = p_0 + p_1 + p_2 = \\ &= \binom{8}{0} 0.31^0 0.69^8 + \binom{8}{1} 0.31^1 0.69^7 + \binom{8}{2} 0.31^2 0.69^6 = \\ &= 0.5264,\end{aligned}$$

b.

$$\begin{aligned}\mathbb{P}(\xi \leqslant 6) &= 1 - \mathbb{P}(\xi > 6) = \\ &= 1 - \mathbb{P}((\xi = 7) \cup (\xi = 8)) = 1 - (p_7 + p_8) = \\ &= 1 - \left(\binom{8}{7} 0.31^7 0.69^1 + \binom{8}{8} 0.31^8 0.69^0\right) = \\ &= 0.9984\end{aligned}$$

Solution to Problem 1.c

The probability of occurrence of an event A is 0.31. Let ξ denote the number of occurrences of event A.

c.

$$\begin{aligned}\mathbb{P}(\xi \geqslant 6) &= \mathbb{P}((\xi = 6) \cup (\xi = 7) \cup (\xi = 8)) = p_6 + p_7 + p_8 = \\ &= \binom{8}{6} 0.31^6 0.69^2 + \binom{8}{7} 0.31^7 0.69^1 + \binom{8}{8} 0.31^8 0.69^0 = \\ &= 0.0134.\end{aligned}$$

In the sequel the part of, for example, $\mathbb{P}((\xi = 6) \cup (\xi = 7) \cup (\xi = 8))$ does not need to be written.

Problem 2

For the final round of an application for a foreign scholarship, 8 candidates of equal ability remained, 5 men and 3 women. The committee then randomly selected 4 of them. What is the probability that there will be a woman among the selected?

Solution to Problem 2

Let ξ denote the number of selected women. Then ξ is a hypergeometric random variable with parameters $N = 8$, $s = 3$, $N - s = 5$, $n = 4$.

$$\mathbb{P}(\text{there is a woman}) = 1 - \mathbb{P}(\text{no woman}) = 1 - p_0 =$$

$$= 1 - \frac{\binom{3}{0}\binom{5}{4}}{\binom{8}{4}} = \frac{13}{14} = 0.9286.$$

Problem 3

In a puzzle contest, 4 prizes are drawn among the correct solutions. (One solver can win at most one prize.) A total of 50 correct solutions were submitted, 20 of them from Miskolc. What is the probability that there will be a winner from Miskolc?

Solution to Problem 3

Let ξ denote the number of selected solvers from Miskolc. Then ξ is a hypergeometric random variable with parameters $N = 50$, $s = 20$, $N - s = 30$, $n = 4$.

$$\mathbb{P}(\text{there is a winner from Miskolc}) = 1 - p_0 = 1 - \frac{\binom{20}{0}\binom{30}{4}}{\binom{50}{4}} = 0.8810.$$

2. Geometric and Negative Binomial Distributions

Problem 4

We perform a sequence of tosses with a fair coin. The fair coin means that the probability of heads $= \frac{1}{2}$, and the probability of tails $= \frac{1}{2}$.

a. On average, how many tosses are needed to get a head for the first time?

b. On average, how many tosses are needed to get a head for the third time?

Solution to Problem 4

a. $\xi \sim \mathrm{Geom}\left(p = \frac{1}{2}\right)$, hence $\mathbb{E}(\xi) = \frac{1}{p} = \frac{1}{\frac{1}{2}} = 2$;

b. $\xi \sim \mathcal{NB}\left(r = 3, p = \frac{1}{2}\right)$, hence $\mathbb{E}(\xi) = \frac{r}{p} = 6$.

3. Poisson Distribution

Problem 5

On average 5 cars arrive for repair in a workshop per shift, and their number follows a Poisson distribution. What is the probability that on one shift at least 4 but at most 7 cars are repaired?

Solution to Problem 5

Let ξ denote the number of cars arriving to the workshop in one shift.
Then $\xi \sim$ Poiss $(\lambda = 5)$. Since the union is disjoint, then by the finite additivity of the probability we obtain that

$$\mathbb{P}(4 \leqslant \xi \leqslant 7) = \mathbb{P}((\xi = 4) \cup (\xi = 5) \cup (\xi = 6) \cup (\xi = 7)) =$$
$$= p_4 + p_5 + p_6 + p_7 = \left(\frac{5^4}{4!} + \frac{5^5}{5!} + \frac{5^6}{6!} + \frac{5^7}{7!}\right) e^{-5} =$$
$$= 0.6016.$$

Problem 6

In a slice of cake the number of raisins follows a Poisson distribution with an average of 9 raisins per slice. What is the probability that in one slice there are at least 7 but at most 10 raisins?

Solution to Problem 6

Let ξ denote the number of raisins in one slice of cake.
Then $\xi \sim$ Poiss $(\lambda = 9)$. Since the union is disjoint, applying finite additivity of probability we obtain that

$$\mathbb{P}(7 \leqslant \xi \leqslant 10) = \mathbb{P}((\xi = 7) \cup (\xi = 8) \cup (\xi = 9) \cup (\xi = 10)) =$$
$$= p_7 + p_8 + p_9 + p_{10} = \left(\frac{9^7}{7!} + \frac{9^8}{8!} + \frac{9^9}{9!} + \frac{9^{10}}{10!}\right) e^{-9} =$$
$$= 0.4992.$$

Problem 7

In one kilogram of cake, there are on average 52 raisins. In a 50 g slices the number of raisins follows a Poisson distribution. At least how many slices should we take so that the probability of having at least one slice without raisins is at least 0.92?

Solution to Problem 7

Let ξ denote the number of raisins in a 50 g slice. Since in 1000 g of cake there are 52 raisins, in 50 g of cake there are $\frac{50 \cdot 52}{1000} = 2.6$ raisins. Thus $\xi \sim \text{Poiss}\,(\lambda = 2.6)$.

Let A_n be the event that in n slices there is at least one without raisins. We look for n such that $\mathbb{P}(A_n) \geqslant 0.92$. Then

$$\mathbb{P}\left(\overline{A_n}\right) = 1 - \mathbb{P}(A_n) \leqslant 1 - 0.92 = 0.08,$$

Here $\overline{A_n}$ means the event that in n slices there is no slice without raisins, i.e. each contains raisins. Using the independence, we get that

$$\mathbb{P}\left(\overline{A_n}\right) = (1 - p_0)^n = \left(1 - \frac{2.6^0}{0!} e^{-2.6}\right)^n = (1 - e^{-2.6})^n \leqslant 0.08.$$

Continuation

The obtained exponential inequality can be solved using logarithms.

$$n \geqslant \frac{\ln(0.08)}{\ln(1 - e^{-2.6})} = 32.7266,$$

thus at least 33 slices must be taken.

End of Seminar 5