Probability and Mathematical Statistics

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Seminar 4

Expected Value and Variance of Random Variable,

1. Expected value, and variance of discrete random variables

Problem 1

A careless student can take the exam at most 4 times, and succeeds at each attempt with probability 0.25. On average, how many times does a careless student take the exam?

Solution to Problem 1

Let ξ denote the number of required exams. Then the random variable ξ can take the values $1, 2, 3, 4$ with the following probabilities

$$\begin{aligned} x_1 &= 1 \qquad & p_1 &= 0.25; \\ x_2 &= 2 \qquad & p_2 &= 0.75 \cdot 0.25; \\ x_3 &= 3 \qquad & p_3 &= 0.75^2 \cdot 0.25; \\ x_4 &= 4 \qquad & p_4 &= 0.75^3; \end{aligned}$$

Thus

$$\mathbb{E}(\xi) = \sum x_i p_i = 1 \cdot 0.25 + 2 \cdot 0.75 \cdot 0.25 + 3 \cdot 0.75^2 \cdot 0.25 + 4 \cdot 0.75^3 =$$
$$= \frac{175}{64} = 2.7344.$$

Problem 2

We toss a fair die 100 times. What are the expected value and the variance of the sum of the rolled numbers?

Solution to Problem 2

We toss one die once. Let ξ denote the rolled number.

$$\mathbb{E}(\xi) = \frac{1}{6}(1+2+3+4+5+6) = \frac{21}{6} = 3.5,$$

$$\mathbb{E}(\xi^2) = \frac{1}{6}\left(1^2+2^2+3^2+4^2+5^2+6^2\right) = \frac{1}{6}(1+4+9+16+25+36) =$$

$$= \frac{91}{6} = 15.1666,$$

$$\mathbb{D}^2(\xi) = \mathbb{E}(\xi^2) - (\mathbb{E}(\xi))^2 = \frac{91}{6} - \left(\frac{21}{6}\right)^2 = \frac{35}{12} = 2.9126.$$

Let ξ_i denote the number rolled in the i-th roll ($i = 1, 2, \ldots, 100$). Then

$$\mathbb{E}(\xi_1+\xi_2+\cdots+\xi_{100}) = \mathbb{E}(\xi_1)+\mathbb{E}(\xi_2)+\cdots+\mathbb{E}(\xi_{100}) = 100\mathbb{E}(\xi) = 100\cdot\frac{21}{6} = 350.$$

Since the random variables $\xi_1, \xi_2, \ldots, \xi_{100}$ are pairwise independent, we obtain that

$$\mathbb{D}^2(\xi_1+\xi_2+\cdots+\xi_{100}) = \mathbb{D}^2(\xi_1)+\mathbb{D}^2(\xi_2)+\cdots+\mathbb{D}^2(\xi_{100}) = 100\mathbb{D}^2(\xi) =$$

$$= 100\cdot\frac{35}{12} = \frac{875}{30} = 291.6667,$$

$$\mathbb{D}(\xi_1+\xi_2+\cdots+\xi_{100}) = \sqrt{\mathbb{D}^2(\xi)} = \sqrt{291.6667} = 17.0783.$$

Exercise 3

A player and the bank spin a fair spinner marked with the numbers $1, 2, 3, 4, 5, 6, 7$. The player wins if he spins a higher number than the bank. In all other cases the bank wins. If the player wins, he receives 2000 HUF. How much money must the player pay to the bank before the game so that the bank's average profit per game is 200 HUF?

Solution to Exercise 3

Let ξ denote the bank's profit per spin. The random variable ξ can take the values x, $x - 2000$, where x is the amount in HUF that the player must pay to the bank before spinning.
The bank's chances of winning:

P/B	1	2	3	4	5	6	7
1	$*$	$*$	$*$	$*$	$*$	$*$	$*$
2		$*$	$*$	$*$	$*$	$*$	$*$
3			$*$	$*$	$*$	$*$	$*$
4				$*$	$*$	$*$	$*$
5					$*$	$*$	$*$
6						$*$	$*$
7							$*$

Continuation

$$\mathbb{P}(\text{bank wins}) = \frac{\frac{7^2-7}{2}+7}{49} = \frac{4}{7}$$

$$\mathbb{P}(\text{bank loses}) = \frac{3}{7}$$

Thus the distribution of ξ is

x_i	x	x-2000
p_i	$\frac{4}{7}$	$\frac{3}{7}$

The expected value of ξ

$$200 = \mathbb{E}(\xi) = x \cdot \frac{4}{7} + (x - 2000) \cdot \frac{3}{7}$$

$$x = \frac{7400}{7} = 1057.1 \quad \text{(HUF)}$$

Thus, in order for the bank to have an average profit of 200 HUF per spin, the player must pay 1057.1 HUF before each spin for the chance to win 2000 HUF.

2. Expected Value and Variance of Absolutely Continuous Random Variables

Exercise 4

Let ξ be a random variable with density function

$$f(x) = \begin{cases} C(-x^2+1), & \text{if } -1 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Determine the value of the parameter C, the expected value, the median, and the variance of ξ, as well as the probability $\mathbb{P}(\xi < \mathbb{E}(\xi))$.

Solution to Exercise 4

- **Determination of the value of the normalizing constant:**

$$1 = \int_{-\infty}^{\infty} f(x)dx = C\int_{-1}^{1}(-x^2+1)dx = C\left[-\frac{x^3}{3}+x\right]_{x=-1}^{x=1} =$$

$$= C\left(\left(-\frac{1}{3}+1\right)-\left(\frac{1}{3}-1\right)\right) = \frac{4}{3}C \quad \Longrightarrow$$

$$C = \frac{3}{4}.$$

- **Determination of the expected value and the median:**
  Since the density function f is symmetric with respect to the $y=0$ axis, and the expected value exists, both the expected value and the median are equal to 0. ($\mathbb{E}(\xi)=0$, $\nu=0$.)

Continuation

- **Determination of the variance:**
  First, we calculate the second moment, which, since $\mathbb{E}(\xi) = 0$, will also equal to the variance.

$$\mathbb{D}^2(\xi) = \mathbb{E}(\xi^2) - [\mathbb{E}(\xi)]^2 = \mathbb{E}(\xi^2) =$$

$$= \frac{3}{4} \int_{-1}^{1} x^2(-x^2+1)dx = \frac{3}{4} \int_{-1}^{1} (-x^4+x^2)dx$$

$$= \frac{3}{4} \left[-\frac{x^5}{5} + \frac{x^3}{3} \right]_{x=-1}^{x=1} = \frac{3}{4} \left(\left(-\frac{1}{5} + \frac{1}{3} \right) - \left(\frac{1}{5} - \frac{1}{3} \right) \right) =$$

$$= \frac{3}{4} \cdot 2 \cdot \left(-\frac{1}{5} + \frac{1}{3} \right) = \frac{1}{5} = 0.2.$$

Exercise 5

Let ξ be an absolutely continuous random variable with distribution function

$$\mathbb{F}(x) = \begin{cases} 0, & \text{if } x < 0, \\ C(x^2 + 2x), & \text{if } 0 \leqslant x < 2, \\ 1, & \text{if } x \geqslant 2. \end{cases}$$

Determine the value of the parameter C, the expected value and the value of $\mathbb{E}(29\xi - 680)$.

Solution to Exercise 5

- **Determination of the value of the constant C:** Since the distribution function of an absolutely continuous random variable is absolutely continuous (hence it is continuous), to determine C it suffices to see that the distribution function connects continuously at its breakpoints, thus

$$1 = \mathbb{F}(2) = C(x^2 + 2x)\big|_{x=2} = 8C \quad \Longrightarrow \quad C = \frac{1}{8}.$$

- **Determination of the density function:** The density function f is obtained by differentiating the distribution function $\mathbb{F}(x)$ with respect to x, thus

$$f(x) = \begin{cases} \frac{1}{8}(2x+2) = \frac{1}{4}(x+1), & \text{if } 0 \leqslant x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

Continuation

- **Determination of the expected value:**

$$\mathbb{E}(\xi) = \int_{-\infty}^{\infty} xf(x)dx = \frac{1}{4}\int_0^2 x(x+1)dx = \frac{1}{4}\int_0^2 (x^2+x)dx =$$
$$= \frac{1}{4}\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_{x=0}^{x=2} = \frac{1}{4}\left(\frac{8}{3} + \frac{4}{2}\right) =$$
$$= \frac{7}{6} = 1.6667.$$

- **To determine** $\mathbb{E}(29\xi - 680)$, we use the linearity of expectation, hence

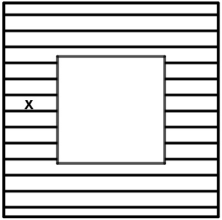
$$\mathbb{E}(29\xi - 680) = 29\mathbb{E}(\xi) - 680 = 29 \cdot \frac{7}{6} - 680 = -\frac{3877}{6} = -646.1667.$$

Exercise 6

We drop a point randomly on a unit square. Let ξ denote the distance of the dropped point from the boundary of the unit square. Determine the distribution function, density function, expected value, and variance of ξ.

Solution to Exercise 6

Let ξ denote the distance of the dropped point from the boundary of the unit square.



Continuation

1. **Determination of the distribution function.**
The range of the random variable ξ is the interval $[0, \frac{1}{2}]$. From this, it immediately follows that:
If $x > \frac{1}{2}$, then $\mathbb{F}(x) = 1$.
If $x \in [0, \frac{1}{2}]$, then

$$\mathbb{F}(x) = \mathbb{P}(\xi < x) = 1 - (1 - 2x)^2 = 4(-x^2 + x) = -4x^2 + 4x$$

If $x < 0$, then clearly $\mathbb{F}(x) = \mathbb{P}(\xi < x) = 0$.
Thus we obtain

$$\mathbb{F}(x) = \begin{cases} 0, & \text{if } x < 0, \\ -4x^2 + 4x, & \text{if } x \in [0, \frac{1}{2}], \\ 1, & \text{if } x > \frac{1}{2}. \end{cases}$$

Continuation

2. **Determination of the density function:**
   The density function can be obtained by differentiating the distribution function. Thus we get that

$$f(x) = \begin{cases} -8x + 4, & \text{if } x \in \left[0, \frac{1}{2}\right], \\ 0, & \text{otherwise.} \end{cases}$$

Continuation

3. Determination of the expected value and the second moment.

$$\mathbb{E}(\xi) = \int_{-\infty}^{\infty} xf(x)dx = \int_0^{\frac{1}{2}} x(-8x+4)dx = \int_0^{\frac{1}{2}} (-8x^2+4x)dx =$$

$$= \left[-\frac{8x^3}{3} + \frac{4x^2}{2}\right]_{x=0}^{x=\frac{1}{2}} = \left(-\frac{1}{3} + \frac{1}{2}\right) = \frac{1}{6},$$

$$\mathbb{E}(\xi^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^{\frac{1}{2}} x^2(-8x+4)dx = \int_0^{\frac{1}{2}} (-8x^3+4x^2)dx =$$

$$= \left[-\frac{8x^4}{4} + \frac{4x^3}{3}\right]_{x=0}^{x=\frac{1}{2}} = -\frac{1}{8} + \frac{1}{6} = \frac{1}{24}.$$

4. Determination of the variance:

$$\mathbb{D}^2(\xi) = \mathbb{E}(\xi^2) - (\mathbb{E}(\xi))^2 = \frac{1}{24} - \left(\frac{1}{6}\right)^2 = \frac{1}{72} = 0.0139.$$

End of Seminar 4