Probability Theory and Mathematical Statistics

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Dr. Tamás Glavosits

Seminar 3.

Bayes' Formula, Law of Total Probability, Bayes' Theorem, Pairwise and Mutual Independence of Events

1. Bayes' Formula, Complete System of Events, Bayes' Theorem

Problem 1

A product is manifactured on 3 machines. We randomly choose one product. Consider the following events:
$B_i :=$ the product is manufactured on the i-th machine ($i = 1, 2, 3$);
$A :=$ a product is defective.
It is also known that $\mathbb{P}(B_1) = 0.1$, $\mathbb{P}(B_2) = 0.2$, $\mathbb{P}(B_3) = 0.7$; and the defect rates of the machines are $0.02, 0.01, 0.005$.

a. What is the probability that a randomly selected product is defective?

b. If a defective product is selected, what is the probability that it was produced by the first, the second, or the third machine?

Solution to Problem 1

Using the law of total probability and Bayes' theorem:

a. $\mathbb{P}(A) = \mathbb{P}(A|B_1)\mathbb{P}(B_1) + \mathbb{P}(A|B_2)\mathbb{P}(B_2) + \mathbb{P}(A|B_3)\mathbb{P}(B_3) = 0.1 \cdot 0.04 + 0.2 \cdot 0.01 + 0.7 \cdot 0.005 = = 0.0095.$

b.

$$\mathbb{P}(B_1|A) = \frac{\mathbb{P}(A|B_1)\mathbb{P}(B_1)}{\mathbb{P}(A)} = \frac{0.1 \cdot 0.04}{0.0095} = \frac{8}{19} = 0.4211$$

$$\mathbb{P}(B_2|A) = \frac{\mathbb{P}(A|B_2)\mathbb{P}(B_2)}{\mathbb{P}(A)} = \frac{0.2 \cdot 0.01}{0.0095} = \frac{4}{19} = 0.2105$$

$$\mathbb{P}(B_3|A) = \frac{\mathbb{P}(A|B_3)\mathbb{P}(B_3)}{\mathbb{P}(A)} = \frac{0.7 \cdot 0.005}{0.0095} = \frac{7}{19} = 0.3684.$$

Problem 2

There are 3 boxes, each contains 4 balls.
- Box 1: 1 red and 3 white,
- Box 2: 2 red and 2 white,
- Box 3: 3 red and 1 white.

We randomly select one box, and from the selected box draw a ball.
What is the probability, that the drawn ball is red?

Solution to Problem 2

Define the following events:
$B_i :=$ the i-th box is chosen $(i = 1, 2, 3)$.
$A :=$ a red ball is drawn.
Since B_1, B_2, B_3 form a complete system of events, by the law of total probability we obtain that

$$\begin{aligned}\mathbb{P}(A) &= \mathbb{P}(A|B_1)\mathbb{P}(B_1) + \mathbb{P}(A|B_2)\mathbb{P}(B_2) + \mathbb{P}(A|B_3)\mathbb{P}(B_3) \\ &= \frac{1}{3}\left(\frac{1}{4} + \frac{2}{4} + \frac{3}{4}\right) = \frac{1}{3} \cdot \frac{6}{4} = \frac{1}{2}.\end{aligned}$$

Problem 3

70% of patients undergo a new treatment, which improves the recovery rate from 80% to 85%. If a recovered patient is selected, what is the probability that he/she received the new treatment?

Solution to Problem 3

Define the following events:
A = a patient recovers;
B = a patient received the new treatment.
We know that

$$\mathbb{P}(B) = 0.7, \quad \mathbb{P}(\overline{B}) = 0.3, \quad \mathbb{P}(A|B) = 0.85, \quad \mathbb{P}(A|\overline{B}) = 0.80.$$

We want to know the conditional probability $\mathbb{P}(B|A)$.
By Bayes' theorem:

$$\begin{aligned}\mathbb{P}(B|A) &= \frac{\mathbb{P}(A|B)\mathbb{P}(B)}{\mathbb{P}(A|B)\mathbb{P}(B) + \mathbb{P}(A|\overline{B})\mathbb{P}(\overline{B})} \\ &= \frac{0.85 \cdot 0.7}{0.85 \cdot 0.7 + 0.80 \cdot 0.3} = \frac{119}{167} = 0.7126.\end{aligned}$$

Problem 4

A mouse can reach the cheese through 3 corridors. Each corridor has 3 doors, and every door is independently open with probability 60%. The doors remain fixed in either the open or closed state. If the mouse sees a corridor in which all doors are open, it will run through to the cheese. What is the probability that the mouse can reach the cheese?

Solution to Problem 4

Let A, B, C denote the events of passing through each corridor. We need $\mathbb{P}(A \cup B \cup C)$. By symmetry:

$$
\begin{aligned}
\mathbb{P}(A \cup B \cup C) &= 3\mathbb{P}(A) - 3\mathbb{P}(A \cap B) + \mathbb{P}(A \cap B \cap C) \\
&= 3 \cdot 0.35^3 - 3 \cdot 0.35^9 = 0.9425.
\end{aligned}
$$

Problem 5

The events X, Y, Z are mutually independent with $\mathbb{P}(X) = \mathbb{P}(Y) = \mathbb{P}(Z) = 0.6$. Define the following events:

A: at most 1 of X, Y, Z occur;

D: at most 2 of X, Y, Z occur;

E: exactly 2 of X, Y, Z occur;

F: at least 2 of X, Y, Z occur.

Solution to Exercise 5

The events X, Y, Z are completely independent,
$\mathbb{P}(X) = \mathbb{P}(Y) = \mathbb{P}(Z) = 0.6$.
Using the results of a previous exercise, we obtain

$$\mathbb{P}(A) = \mathbb{P}(\text{at most } 1) = 1 - \mathbb{P}\left((X \cap Y) \cup (X \cap Z) \cup (Y \cap Z)\right) =$$
$$= 1 - [3\mathbb{P}(X \cap Y) - 3\mathbb{P}(X \cap Y \cap Z) + (x \cap Y \cap Z)] =$$
$$= 1 - \left[3 \cdot 0.6^2 - 20.6^3\right] = \frac{44}{125} = 0.352.$$
$$\mathbb{P}(D) = \mathbb{P}(\text{at most } 2) = \mathbb{P}(\overline{X \cap Y \cap Z}) = 1 - \mathbb{P}(X)^3 =$$
$$= 1 - 0.6^3 = \frac{98}{125} = 0.784.$$
$$\mathbb{P}(E) = \mathbb{P}(\text{exactly } 2) = 3\mathbb{P}(X \cap Y \cap \overline{Z}) = 3\mathbb{P}(X)^2(1 - \mathbb{P}(X)) =$$
$$= 3 \cdot 0.6^2 \cdot 0.4 = \frac{54}{125} = 0.432.$$
$$\mathbb{P}(F) = \mathbb{P}(\text{at least } 2) = 1 - \mathbb{P}(A) = 1 - \frac{44}{125} = \frac{81}{125} = 0.648.$$

Problem 6

Two players shoot a basketball. Player A scores with probability 0.6, player B with probability 0.8. The game continues until one of them scores. What is the probability that:

a. A starts and A wins;

a. A starts and B wins;

a. B starts and A wins;

a. B starts and B wins.

Solution to Exercise 6

(We assume that the shots are independent.)

a. • We are going to solve the problem parametrically.
Let us introduce the following parameters.

$$\mathbb{P}(A \text{ scores }) = p, \qquad \mathbb{P}(B \text{ scores }) = q.$$

Suppose A starts and we want to know the probability that A wins. It is obvious that player A can win on the 1st, 3rd, 5th, ... shots.

First shot $\mathbb{P}(A \text{ scores }) = p$;
Third shot $\mathbb{P}(A \text{ scores }) = (1-p)(1-q)p$;
Fifth shot $\mathbb{P}(A \text{ scores }) = [(1-p)(1-q)]^2 p$;
$(\dots)$

Continuation of the solution to Exercise 6 a)

$$\mathbb{P}(A \text{ starts and } A \text{ wins}) =$$

$$= p + (1-p)(1-q)p + [(1-p)(1-q)]^2 p + \cdots =$$

$$= p \sum_{k=0}^{\infty} [(1-p)(1-q)]^k = \frac{p}{1-(1-p)(1-q)}.$$

In the special case of $p = 0.6$, $q = 0.8$, we get that

$$\mathbb{P}(A \text{ starts and } A \text{ wins}) = \frac{0.6}{1 - 0.4 \cdot 0.2} = \frac{15}{23} = 0.6522.$$

Solution to Exercise 6 b)

b. A starts and B wins

$$\mathbb{P}(A \text{ starts and } B \text{ wins}) = 1 - \mathbb{P}(A \text{ starts and } A \text{ wins}) = \\ = 1 - \frac{p}{1-(1-p)(1-q)}.$$

In the special case of $p = 0.6$, $q = 0.8$, we get that

$$\mathbb{P}(A \text{ starts and } B \text{ wins}) = 1 - \frac{15}{23} = \frac{8}{23} = 0.3478.$$

Solution to Exercise 6 c)

c. B starts and A wins. By the above, it is easy to see that

$$\mathbb{P}(B \text{ starts and } A \text{ wins}) = 1 - \frac{q}{1-(1-p)(1-q)}.$$

In the special case of $p = 0.6$, $q = 0.8$, we get that

$$\mathbb{P}(B \text{ starts and } A \text{ wins}) = 1 - \frac{0.8}{1 - 0.4 \cdot 0.2} = \frac{3}{23} = 0.1304.$$

Solution to Exercise 6 d)

d. B starts and B wins. By the above, it is easy to see that

$$\mathbb{P}(B \text{ starts and } B \text{ wins}) = \frac{q}{1-(1-p)(1-q)}.$$

In the special case of $p = 0.6$, $q = 0.8$, we get that

$$\mathbb{P}(B \text{ starts and } B \text{ wins}) = \frac{0.8}{1-0.4\cdot 0.2} = \frac{20}{23} = 0.8696.$$

End of Seminar 3