Probability and Mathematical Statistics

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Seminar 2

Drawing Balls from an Urn, Geometric Probability, Operations with Events,

1. Drawing Balls from an Urn with and without Replacement

Problem 1

An urn contains only red and white balls, 10 red and 20 white balls. Without replacement, we draw 10 balls. What is the probability that the number of red balls drawn is at most 3?

Solution to Problem 1

$$\mathbb{P}(\xi \geqslant 3) = 1 - \mathbb{P}(\xi < 3) = 1 - (p_0 + p_1 + p_2) =$$

$$= 1 - \frac{\binom{10}{0}\binom{20}{10} + \binom{10}{1}\binom{20}{9} + \binom{10}{2}\binom{20}{8}}{\binom{30}{10}} =$$

$$= 0.7493.$$

Problem 2

An urn contains only red and white balls, 1 red and 2 white balls. With replacement, we draw 4 balls. What is the probability that the number of red balls drawn is at most 2?

Solution to Problem 2

Let ξ denote the number of red balls drawn. Then $\xi \sim \mathcal{B}(p = \frac{1}{3}, n = 4)$. The desired probability is:

$$\mathbb{P}(\xi \leqslant 2) = p_0 + p_1 + p_2 =$$

$$= \binom{4}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^4 + \binom{4}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^3 + \binom{4}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2 =$$

$$= \frac{56}{81} = 0.6914.$$

Problem 3

An urn contains only red and white balls, 10 red and 20 white balls. With replacement, we draw 10 balls. What is the probability that at least 3 of the balls drawn are red?

Solution to Problem 3

Let ξ denote the number of red balls drawn. Then $\xi \sim \mathcal{B}(p = \frac{1}{3}, n = 10)$. The desired probability is:

$$\mathbb{P}(\xi \geqslant 3) = 1 - \mathbb{P}(\xi < 3) = 1 - (p_0 + p_1 + p_2) =$$

$$= 1 - \left(\binom{10}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^{10} + \binom{10}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^9 + \binom{10}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^8\right) =$$

$$= 0.7009.$$

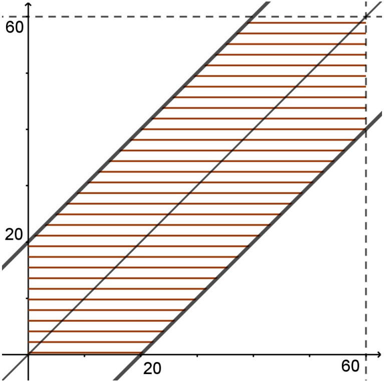
3. Geometric Probability

Problem 4

Two people arrive randomly at a café between 12 am and 1 pm, where they each stay for 20 minutes. What is the probability that they meet?

Solution to Problem 4

$\Omega := [0,60] \times [0,60]$; $A := \{(x,y) \in \Omega \mid |x-y| \leqslant 20\}$.



$$|x-y| \leqslant 20$$
$$\Updownarrow$$
$$-20 \leqslant x-y \leqslant 20$$
$$\Updownarrow$$
$$y \leqslant x+20 \text{ and } y \geqslant x-20$$

$T(\Omega) = 60^2$ and $T(A) = 60^2 - 2 \cdot \frac{40^2}{2} = 60^2 - 40^2$ Thus,

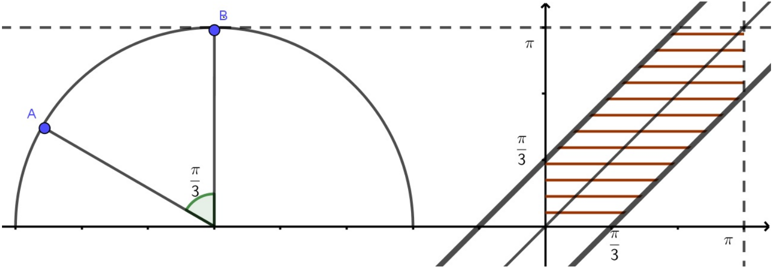
$$\mathbb{P}(A) = \frac{T(A)}{T(\Omega)} = \frac{60^2 - 40^2}{60^2} = 1 - \left(\frac{2}{3}\right)^2 = \frac{5}{9} = 0.4444.$$

Problem 5

Two points A and B are placed randomly on a semicircle. What is the probability that the chord connecting them is less then or equal to the length to the radius of the semicircle?

Solution to Problem 5

$\Omega := [0,\pi] \times [0,\pi]$; $A := \{(x,y) \in \Omega \mid |x-y| \leqslant \frac{\pi}{3}\}$.



Thus, as in the previous problem,

$$\mathbb{P}(A) = \frac{T(A)}{T(\Omega)} = \frac{5}{9} = 0.4444.$$

Problem 6

Two points x and y are chosen randomly on the interval $[0, 1]$. These two points divide $[0, 1]$ into 3 segments. What is the probability that these 3 segments can form a triangle?

Solution to Problem 6

Introduce the notation:

$$\begin{aligned}
\Omega &\doteq [0,1] \times [0,1], \\
A &\doteq \{(x,y) \in \Omega |\ \text{the 3 segments form a triangle}\}, \\
A_1 &\doteq \{(x,y) \in A | x \leqslant y\}, \\
A_2 &\doteq \{(x,y) \in A | y < x\}.
\end{aligned}$$

Then

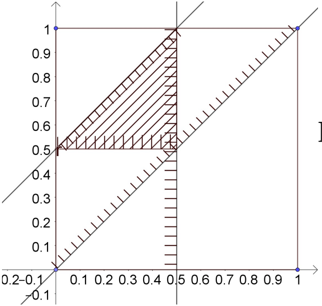
$$A = A_1 \cup A_2, \quad A_1 \cap A_2 = \emptyset, \quad \mathbb{P}(A_1) = \mathbb{P}(A_2), \quad \mathbb{P}(A_1) + \mathbb{P}(A_2) = 2\mathbb{P}(A_1),$$

so it is enough to compute $\mathbb{P}(A_1)$. Assume $x \leqslant y$. The points x and y divide $(0,1)$ into 3 parts.

The lengths of the three segments are: x, $y-x$, $1-y$. Applying the triangle inequality, we get:

$$
\begin{aligned}
x+(y-x) \geqslant 1-y \quad &\Leftrightarrow \quad y \geqslant 1-y \quad \Leftrightarrow \quad 2y \geqslant 1 \quad \Leftrightarrow \quad y \geqslant \frac{1}{2}, \\
x+1-y \geqslant y-x \quad &\Leftrightarrow \quad 2x+1 \geqslant 2y \quad \Leftrightarrow \quad x+\frac{1}{2} \geqslant y, \\
y-x+1-y \geqslant x \quad &\Leftrightarrow \quad 1 \geqslant 2x \quad \Leftrightarrow \quad x \leqslant \frac{1}{2}.
\end{aligned}
$$

The resulting linear inequalities define half-planes in the plane, and we need to intersect these half-planes. We obtain the boundary line of the half-planes by replacing the smaller or larger inequality signs with equality signs. The resulting line divides the plane into two half-planes. We can determine which points of the half-planes satisfy the linear inequality by substituting an arbitrarily chosen point, say the origin, into the linear inequality.



$$\mathbb{P}(A_1) = \frac{t(A_1)}{t(\Omega)} = \frac{\frac{\left(\frac{1}{2}\right)^2}{2}}{1} = \frac{1}{8},$$

$$\Rightarrow \quad \mathbb{P}(A) = 2\mathbb{P}(A_1) = 2 \cdot \frac{1}{8} = \frac{1}{4}.$$

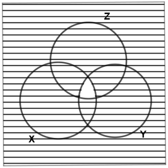
Problem 7

Let X, Y, Z be events, and define the events D, E, F as follows:

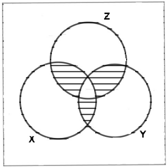
1. D occurs $\iff$ at most 2 of X, Y, Z occur;
2. E occurs $\iff$ exactly 2 of X, Y, Z occur;
3. F occurs $\iff$ at least 2 of X, Y, Z occur.

Solution to Problem 7

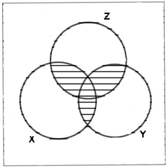
b. •



$$D = \overline{X \cap Y \cap Z}$$



$$E = (X \cap Y \cap \overline{Z}) \cup (X \cap \overline{Y} \cap Z) \cup (\overline{X} \cap Y \cap Z)$$



$$F = (X \cap Y) \cup (X \cap Z) \cup (Y \cap Z)$$

End of Seminar 2