Probability and Mathematical Statistics

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Seminar 10

Statistics I. Point Estimates

1. Basic Statistics

Problem 1

10-element sample:

$$3.6,\ 1.3,\ 0.5,\ 6.2,\ 1.0,\ 9.8,\ 3.0,\ 3.1,\ 6.5,\ 7.6.$$

Determine the

a. mean,

b. empirical second moment,

c. empirical variance,

d. corrected empirical variance,

e. empirical median,

f. median absolute deviation.

Solution of Problem 1

10-element sample:

$$3.6,\ 1.3,\ 0.5,\ 6.2,\ 1.0,\ 9.8,\ 3.0,\ 3.1,\ 6.5,\ 7.6.$$

a. mean: $\overline{\xi} = 4.26$,
b. empirical second moment: $m_2 = 26.9$,
c. empirical variance: $s_n^2 = 8.7524$,
d. corrected empirical variance: $s_n^{*2} = 9.7249$,
e. empirical median: $\mathrm{med} = 3.35$,
f. median absolute deviation: $\mathrm{MAD} = 2.6$.

2. Maximum Likelihood Method

Problem 2

Using the method of moments and the maximum likelihood method, determine

a. the unknown parameter p of a sample from the Bernoulli distribution,
b. the unknown parameter λ of the Poisson distribution,
c. the unknown parameter λ of the exponential distribution,
d. the parameters m and σ^2 of the normal distribution.

Solution of Problem 2. a.

a. **Bernoulli distribution.**

It is easy to compute using the method of moments because if ξ has a Bernoulli distribution with parameter p, then $\mathbb{E}(\xi) = p$. Since $\mathbb{E}(\xi) \sim \overline{\xi}$, we have $\widehat{p} = \overline{\xi}$.

The maximum likelihood method requires a few more steps.

- **Distribution:** $\mathbb{P}(\xi = k) = p^k(1-p)^{1-k}$ $(k \in \{0,1\})$;
- $L(\lambda) = \prod_{i=1}^n p^{\xi_i}(1-p)^{1-\xi_i}$;
- $\frac{\partial}{\partial\lambda}\ln(\lambda) = \sum_{i=1}^n \left(\frac{\xi_i}{p} - \frac{1-\xi_i}{1-p}\right) = \sum_{i=1}^n \frac{(1-p)\xi_i - p(1-\xi_i)}{p(1-p)} = 0.$
  Multiply both sides by $p(1-p)$:
  $0 = \sum_{i=1}^n ((1-p)\xi_i - p(1-\xi_i)) =$
  $\sum_{i=1}^n(\xi_i - p\xi_i - p + p\xi_i) = \sum_{i=1}^n(\xi_i - p) = \sum_{i=1}^n \xi_i - np = 0,$
  from which we get

$$\widehat{p} = \frac{1}{n}\sum_{i=1}^n \xi_i = \overline{\xi},$$

so we get the same result as with the method of moments.

Solution of Problem 2. b.

b. **Poisson distribution.**

It is easy to compute using the method of moments because if $\xi \sim \mathrm{Poiss}(\lambda)$, then $\mathbb{E}(\xi) = \lambda$. Since $\mathbb{E}(\xi) \sim \overline{\xi}$, we get $\widehat{\lambda} = \overline{\xi}$. The likelihood method leads to the same solution.

- $p(k, \lambda) = \frac{\lambda^k}{k!} e^{-\lambda}$ $(k \in 0, 1, 2, \dots)$;
- $L(\lambda) = \prod_{i=1}^{n} \frac{\lambda^{\xi_i}}{\xi_i!} e^{-\lambda}$;
- $l(\lambda) = \log(L(\lambda)) = \sum_{i=1}^{n} (\xi_i \ln(\lambda) - \ln(\xi_i!) - \lambda)$;
- $0 = \frac{\partial}{\partial\lambda} l(\lambda) = \sum_{i=1}^{n} \left(\frac{\xi_i}{\lambda} - 1 \right) = \frac{\sum_{i=1}^{n} \xi_i}{\lambda} - n = 0$;

  from which we get

$$\widehat{\lambda} = \frac{1}{n} \sum_{i=1}^{n} \xi_i = \overline{\xi}.$$

Solution of Problem 2. c.

c. **Exponential distribution.**

It is easy to compute using the method of moments because if $\xi \sim \mathrm{Exp}(\lambda)$, then $\mathbb{E}(\xi) = \frac{1}{\lambda}$. Since $\mathbb{E}(\xi) \sim \overline{\xi}$, we get

$$\widehat{\lambda} = \frac{1}{\overline{\xi}}.$$

The likelihood method leads to the same solution.

- $f(x, \lambda) = \lambda e^{-\lambda x} \ (x > 0)$;
- $L(\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda \xi_i}$;
- $l(\lambda) = \ln(L(\lambda)) = \sum_{i=1}^{n} (\ln(\lambda) - \lambda \xi_i)$;
- $0 = \frac{\partial}{\partial \lambda} l(\lambda) = \sum_{i=1}^{n} \left(\frac{1}{\lambda} - \xi_i\right) = \frac{n}{\lambda} - \sum_{i=1}^{n} \xi_i$, from which we get
$$\widehat{\lambda} = \frac{1}{\frac{1}{n}\sum_{i=1}^{n} \xi_i} = \frac{1}{\overline{\xi}}.$$

Solution of Problem 2. d.

d. **Normal distribution.**

Using the method of moments, if $\xi \sim \mathcal{N}(m, \sigma^2)$, then $\mathbb{E}(\xi) = m$ and $\mathbb{D}^2(\xi) = \sigma^2$. Since $\mathbb{E}(\xi) \sim \overline{\xi}$, we get $\widehat{m} = \overline{\xi}$. Also, $\mathbb{D}^2(\xi) \sim s_n^2$, thus $\sigma^2 \sim s_n^2 = m_2 - \overline{\xi}^2$.

Applying the maximum likelihood method is also straightforward:

- $f(x, m, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}}$ $(x \in \mathbb{R})$, here $\sigma = \sqrt{\sigma^2}$, and we consider σ^2 as the parameter;
- $L(m) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\xi_i - m)^2}{2\sigma^2}}$;
- $l(m) = \ln(L(m)) = \sum_{i=1}^n \left(\ln\left(\frac{1}{\sqrt{2\pi}\sigma}\right) - \frac{(\xi_i - m)^2}{2\sigma^2}\right)$;
- $\frac{\partial}{\partial m} l(m) = \sum_{i=1}^n \frac{1}{2\sigma^2} \cdot 2(\xi_i - m) = \frac{1}{\sigma^2} \sum_{i=1}^n (\xi_i - m) = \frac{1}{\sigma^2} \left(\sum_{i=1}^n \xi_i - nm\right) = 0$, from which we get

$$\widehat{m} = \frac{1}{n} \sum_{i=1}^n \xi_i = \overline{\xi}.$$

2. d. Solution, continuation

Estimation of the unknown parameter σ^2 using the maximum likelihood method:

- $L(\sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sqrt{\sigma^2}} e^{-\frac{(\xi_i - \overline{\xi})^2}{2\sigma^2}}$
- $l(\sigma^2) = \ln(L(\sigma^2)) = \sum_{i=1}^{n} \left(\ln\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{1}{2}\ln(\sigma^2) - \frac{(\xi_i - \overline{\xi})^2}{2\sigma^2} \right);$
- $\frac{\partial}{\partial \sigma^2} l(\sigma^2) = \sum_{i=1}^{n} \left(-\frac{1}{2\sigma^2} + \frac{(\xi_i - \overline{\xi})^2}{2(\sigma^2)^2} \right) = 0 \qquad / \cdot 2\sigma^2$
  thus we get $-\sum_{i=1}^{n} 1 + \frac{1}{\sigma^2}\sum_{i=1}^{n}(\xi_i - \overline{\xi})^2 = 0$, from which it follows that
  $$\widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(\xi_i - \overline{\xi})^2 = s_n^2.$$

Problem 3

Estimation of the unknown parameter ϑ from a sample drawn from the $U(0, \vartheta)$ distribution.

Solution of Problem 3

Let $\xi_1, \xi_2, \ldots, \xi_n$ be an independent sample from the $U(0, \vartheta)$ distribution.

1. Estimator: $2\overline{\xi}$

Indeed, since $\overline{\xi} = \frac{1}{n}\sum_{i=1}^{n} \xi_i$, we have

$$\mathbb{E}(\overline{\xi}) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}(\xi_i) = \frac{1}{n} \cdot n \cdot \frac{\vartheta}{2} = \frac{\vartheta}{2},$$

so $2\overline{\xi}$ is indeed an unbiased estimator.

$$\mathbb{D}^2(2\overline{\xi}) = 4\frac{1}{n^2}\mathbb{D}^2\left(\sum_{i=1}^{n} \xi_i\right) = \frac{4}{n^2} n \frac{\vartheta^2}{12} = \frac{1}{3n}\vartheta^2$$

Solution of Problem 3, continuation

2. Estimator: $\frac{n+1}{n}\max(\xi_1, \xi_2, \dots, \xi_n)$
A theorem regarding $\mathbb{F}_{\max}$ and $\mathbb{F}_{\min}$.
Step 1: If $\xi_1, \xi_2, \dots, \xi_n$ are independent random variables with distribution functions $\mathbb{F}_1, \mathbb{F}_2, \dots, \mathbb{F}_n$, and we define $\eta := \max(\xi_1, \xi_2, \dots, \xi_n)$, then the distribution function of η is

$$\mathbb{F}(x) = \mathbb{F}_1(x)\mathbb{F}_2(x)\dots\mathbb{F}_n(x),$$

because

$$\begin{aligned}\mathbb{F}(x) &= \mathbb{P}(\max(\xi_1, \xi_2, \dots, \xi_n) < x) = \\ &= \mathbb{P}((\xi_1 < x) \cap (\xi_2 < x) \cap \dots \cap (\xi_n < x)) = \\ &= \mathbb{P}((\xi_1 < x)\mathbb{P}(\xi_2 < x)\dots\mathbb{P}(\xi_n < x)) = \mathbb{F}_1(x)\mathbb{F}_2(x)\dots\mathbb{F}_n(x).\end{aligned}$$

Solution of Problem 3, continuation

Step 2: Let $\xi_1, \xi_2, \ldots, \xi_n$ be an independent sample from the $U(0, \vartheta)$ distribution. Then the distribution and density function of $\eta = \max(\xi_1, \xi_2, \ldots, \xi_n)$ are:

$$\mathbb{F}(x) = \begin{cases} 0, \text{ ha } x < 0; \\ \left(\frac{x}{\vartheta}\right)^n, \text{ ha } 0 \leqslant x < \vartheta; \\ 1, \text{ ha } x \geqslant \vartheta. \end{cases}$$

$$f(x) = \begin{cases} \frac{n}{\vartheta}\left(\frac{x}{\vartheta}\right)^{n-1}, \text{ ha } x \in (0\vartheta); \\ 0, \text{ egyébként.} \end{cases}$$

Solution of Problem 3, continuation

The expected value and variance of η are:

$$\mathbb{E}(\eta) = \int_0^{\vartheta} x \frac{x}{\vartheta}\left(\frac{x}{\vartheta}\right)^{n-1} dx = \frac{n}{\vartheta^2}\int_0^{\vartheta} x^n dx =$$

$$= \frac{n}{\vartheta^n}\left[\frac{x^{n+1}}{n+1}\right]_{x=0}^{x=\vartheta} = \frac{n}{\vartheta^n}\frac{\vartheta^{n+1}}{n+1} = \frac{n}{n+1}\vartheta,$$

$$\mathbb{E}(\eta^2) = \int_0^{\vartheta} x^2 \frac{n}{\vartheta}\left(\frac{x}{\vartheta}\right)^{n-1} dx = \frac{n}{\vartheta^n}\int_0^{\vartheta} x^{n+1} dx =$$

$$= \frac{n}{\vartheta^n}\left[\frac{x^{n+2}}{n+2}\right]_{x=0}^{x=\vartheta} = \frac{n}{n+2}\frac{\vartheta^{n+2}}{\vartheta^n} = \frac{n}{n+2}\vartheta^2,$$

$$\mathbb{D}^2(\eta) = \mathbb{E}(\eta^2) - (\mathbb{E}(\eta))^2 = \frac{n}{n+2}\vartheta^2 - \left(\frac{n}{n+1}\vartheta\right)^2 =$$

$$= \frac{2n^2 - n}{(n+2)(n+1)^2}\vartheta^2$$

Solution of Problem 3, continuation

Step 3: The estimator ξ_n^* is not a good estimator since it is biased. The unbiased estimator is $\frac{n+1}{n}\xi_n^*$, i.e., $\frac{n+1}{n}\eta$.
The variance of the estimator is

$$\mathbb{D}^2\left(\frac{n+1}{n}\eta\right) = \frac{(n+1)^2}{n^2}\frac{n(2n-1)}{(n+2)(n+1)^2}\vartheta^2 = \frac{2n-1}{n(n+2)}\vartheta^2.$$

Hence, $2\overline{\xi}$ is the more efficient estimator.

$$\frac{1}{3n} < \frac{2n-1}{n(n+2)} \iff \frac{1}{3} < \frac{2n-1}{n+2} \iff n+2 < 6n-3$$

$$\iff 5 < 6n \iff \frac{5}{6} < n.$$

End of Seminar 10