Probability Theory and Mathematical Statistics

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Lecture 6

Famous Absolutely Continuous Random Variables

1. Mathematical Tools

Characteristic Function of a Random Variable

Definition

The **characteristic function** of an absolutely continuous random variable ξ is the function $\varphi : \mathbb{R} \to \mathbb{C}$ defined by

$$\varphi(t) := \int_{-\infty}^{+\infty} f(x) e^{itx} dx \qquad (t \in \mathbb{R}),$$

where f is the density function of ξ.

Properties of Characteristic Functions

Preserving our notation

Theorem

1. *If* $\mathbb{E}(|\xi|) < +\infty$, *then*
$$\mathbb{E}(\xi) = \frac{\varphi'(0)}{i};$$
2. *If* $\mathbb{E}(|\xi|^2) < \infty$, *then* $\mathbb{E}(\xi^2) = -\varphi''(0)$, *thus*
$$\mathbb{D}^2(\xi) = -\varphi''(0) + \varphi'(0)^2.$$
3. *If* ξ_1 *and* ξ_2 *are independent random variables, then*
$$\varphi_{\xi_1+\xi_2}(t) = \varphi_{\xi_1}(t)\varphi_{\xi_2}(t),$$
that is, the characteristic function converts the sum of independent variables into a product.

Uniqueness Theorem

Theorem

- ***Uniqueness theorem:***

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-itx} \varphi(t) dt,$$

i.e., the characteristic function uniquely determines the absolutely continuous distribution from which it originates.

Method of Moments

It often occurs that a sample comes from a known distribution, but the parameters of distribution are unknown. The unknown parameters can be expressed from the moments (or from sample moments), although often a nonlinear system of equations must be solved. This method is called the **method of moments**.

Convolution of Density Functions

Definition

If $f_1 : \mathbb{R} \to \mathbb{R}$ and $f_2 : \mathbb{R} \to \mathbb{R}$ are integrable functions, then its **convolution** denoted by $f_1 * f_2 : \mathbb{R} \to \mathbb{R}$ is defined by

$$(f_1 * f_2)(x) = \int_{-\infty}^{+\infty} f_1(x - t) f_2(t) dt \qquad (x \in \mathbb{R}),$$

Theorem about the Convolution of Density Functions

Theorem

Let ξ_1 and ξ_2 be independent absolutely continuous random variables with density functions f_1 and f_2.
Then the density function of $\xi_1 + \xi_2$ is

$$f_{\xi_1+\xi_2}(x) = (f_1 * f_2)(x) \qquad (x \in \mathbb{R}),$$

so the density function of the sum of independent absolutely continuous random variables is equal to the convolution of the density functions of the random variables.

2. Notable Absolutely Continuous Random Variables

Uniform Distribution

Definition

A random variable ξ is said to be a **uniform distribution** on interval $[a, b]$ if the probability of falling into a subinterval of $[a, b]$ is proportional to its length denoted by $\xi \sim U(a, b)$.
Its **distribution function and density** function are:

$$\mathbb{F}_\xi(x) = \begin{cases} 0, & \text{if } x < a; \\ \dfrac{x-a}{b-a}, & \text{if } x \in [a, b]; \\ 1, & \text{if } x > b; \end{cases} \qquad f_\xi(x) = \begin{cases} \dfrac{1}{b-a}, & \text{if } x \in [a, b]; \\ 0, & \text{if } x \notin [a, b]. \end{cases}$$

Expectation and Variance of Uniform Distribution

Theorem

*If $\xi \sim U(a, b)$, then its **expectation and variance** are:*

$$\mathbb{E}(\xi) = \frac{a+b}{2}, \qquad \mathbb{D}^2(\xi) = \frac{(b-a)^2}{12}.$$

Exponential Distribution

Definition

Let $\lambda > 0$ be a fixed real constant. A random variable ξ is said to be an **exponential distribution with parameter** λ if its **distribution and density function** are:

$$\mathbb{F}_\xi(x) = \begin{cases} 0, & \text{if } x < 0; \\ 1 - e^{-\lambda x}, & \text{if } x \geq 0; \end{cases} \quad f_\xi(x) = \begin{cases} 0, & \text{if } x < 0; \\ \lambda e^{-\lambda x}, & \text{if } x \geq 0. \end{cases}$$

Denoted by $\xi \sim Exp(\lambda)$.

Expectation and Variance of Exponential Distribution

Theorem

If $\xi \sim Exp(\lambda)$, then its ***expectation and variance*** *are:*

$$\mathbb{E}(\xi) = \frac{1}{\lambda}, \qquad \mathbb{D}^2(\xi) = \frac{1}{\lambda^2}.$$

Properties of Exponential Distribution

Theorem

1. ***Memoryless property:*** *If* $\xi \sim \mathrm{Exp}(\lambda)$, *then*

$$\mathbb{P}(\xi < s + t | \xi > s) = \mathbb{P}(\xi < t) \qquad (s, t > 0).$$

2. ***Poisson process:*** *If the number of events per unit time follows a Poisson distribution, then the time between two successive events is an exponential distribution with the same parameter* λ.

Normal Distribution

Definition

A random variable η is said to be a **standard normal distribution** (denoted by $\eta \sim \mathcal{N}(0,1)$) if its **distribution and density function** are:

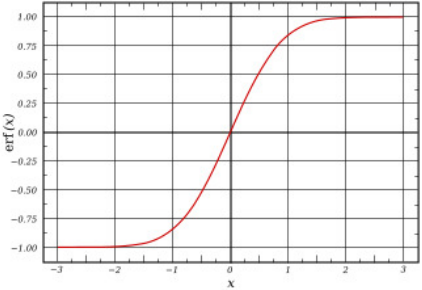
$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt, \qquad \varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},$$

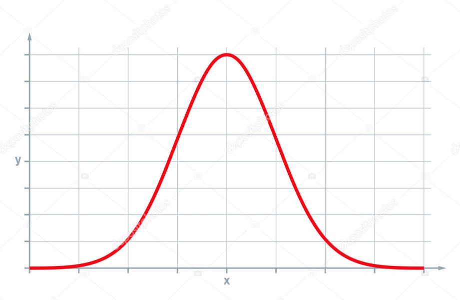
for any $x \in \mathbb{R}$.

Remark

The funciton Φ is said to be Gaussion bell curve.

The distribution and desity function of the standard normal distribution





Distribution and Variance of Standard Normal

Theorem

If $\eta \sim \mathcal{N}(0,1)$, *then*

$$\mathbb{E}(\eta) = 0, \qquad \mathbb{D}^2(\eta) = 1.$$

Normal Random Variable with Parameters m and σ^2

Definition

A random variable ξ is said to be a **normal distribution with parameters** m**, and** σ^2 (denoted by $\xi \sim \mathcal{N}(m, \sigma^2)$) if there exists $\eta \sim \mathcal{N}(0, 1)$ such that $\xi = \sigma\eta + m$, with $\sigma > 0$, $m \in \mathbb{R}$.

Distribution Function, Density function, Expectation and Variance of $\xi \sim \mathcal{N}(m, \sigma^2)$

Theorem

If $\xi \sim \mathcal{N}(m, \sigma^2)$, then

- ***Distribution and density function*** *for all $x \in \mathbb{R}$:*

$$\mathbb{F}(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{-\frac{(t-m)^2}{2\sigma^2}} dt, \qquad f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}};$$

- ***Expectation and variance:***

$$\mathbb{E}(\xi) = m, \qquad \mathbb{D}^2(\xi) = \sigma^2.$$

Properties

Theorem

1. ***Standardization:*** *If* $\xi \sim \mathcal{N}(m, \sigma^2)$, *then* $\eta \doteq \frac{\xi - m}{\sigma} \sim \mathcal{N}(0, 1)$.
2. $\Phi(-x) = 1 - \Phi(x)$ *for all* $x \in \mathbb{R}$.
3. *If* $\xi_1 \sim \mathcal{N}(m_1, \sigma_1^2)$, $\xi_2 \sim \mathcal{N}(m_2, \sigma_2^2)$ *are independent random variable, then* $\xi_1 + \xi_2 \sim \mathcal{N}(m_1 + m_2, \sigma_1^2 + \sigma_2^2)$.

3. Further Distributions Derived from the Standard Normal Distribution

χ^2 Distribution with n Degrees of Freedom

Definition

Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent standard normal random variables. Then the random variable

$$\eta_n := \xi_1^2 + \xi_2^2 + \cdots + \xi_n^2$$

is called the χ^2 **distribution with** n **degrees of freedom**. Denoted by $\eta_n \sim \chi_n^2$.

t Distribution with n Degrees of Freedom

Definition

Let $\xi \sim \mathcal{N}(0,1)$ and $\eta_n \sim \chi^2_n$ be independent. Then

$$\zeta_n := \frac{\xi}{\sqrt{\frac{\eta_n}{n}}}$$

is called the t **distribution with** n **degrees of freedom (Student's** t **distribution)**. Denoted by $\zeta_n \sim t_n$.

F Distribution

Definition

Let $\xi_n \sim \chi_n^2$ and $\eta_m \sim \chi_m^2$ be independent. Then

$$\zeta_{n,m} = \frac{\frac{\xi_n}{n}}{\frac{\eta_m}{m}}$$

is called the F **distribution (Fisher-Snedecor distribution)** with n and m degrees of freedom. Denoted by $\zeta_{n,m} \sim F(d_1, d_2)$ ($d_1 = n$, $d_2 = m$ indicate the degrees of freedom).

End of Lecture 6