Probability Theory and Mathematical Statistics

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Lecture 4

Random Variables

1. Random Variables

Random Variables

Definition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A function $\xi : \Omega \to \mathbb{R}$ is called **random variable** if

$$(\xi < x) := \{\omega \in \Omega \mid \xi(\omega) < x\} = \xi^{-1}\left(]-\infty, x[\right) \in \mathcal{F}$$

for all $x \in \mathbb{R}$, that is the level sets of ξ are events.

Theorem

If ξ is a random variable, then

$$(\xi = x) := \{\omega \in \Omega \mid \xi(\omega) = x\} = \xi^{-1}(x) \in \mathcal{F}$$

for all $x \in \mathbb{R}$.

Distribution Function of a Random Variable

Definition

The **distribution function** of a random variable ξ is the function $\mathbb{F}_\xi : \mathbb{R} \to \mathbb{R}$ defined by

$$\mathbb{F}_\xi(x) := \mathbb{P}(\xi < x) \qquad (x \in \mathbb{R}).$$

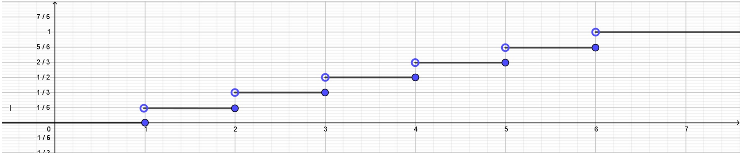
Remark

Every random variable has a distribution function.

Example: Distribution Function of a Random Variable

We toss a die. This experiment is described by the classical probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\Omega = \{1, 2, 3, 4, 5, 6\}$. Let ξ denote the outcome. Then the distribution function of ξ is

$$\mathbb{F}(x) = \begin{cases} 0, & \text{if } x \in]-\infty, 1[, \\ \dfrac{k}{6}, \quad \text{if } x \in]k-1, k], & (k = 2, 3, 4, 5, 6), \\ 1, & \text{if } x > 6. \end{cases}$$



Which is a monotone increasing step-function with $\lim_{x \to -\infty} \mathbb{F}(x) = 0$, and $\lim_{x \to +\infty} \mathbb{F}(x) = 1$

Properties of the Distribution Function

Theorem

Let ξ be a random variable with distribution function $\mathbb{F}$. Then $\mathbb{F}$ has the following properties:

1. *Monotone increasing;*
2. *Left-continuous;*
3. $\lim_{x \to -\infty} \mathbb{F}(x) = 0, \ \lim_{x \to +\infty} \mathbb{F}(x) = 1.$

2. Discrete Random Variables

Distribution of Discrete Random Variables

A sequence (x_n) is called **injective** if $x_i \neq x_j$ whenever $i \neq j$.

Definition

A random variable ξ is called a **discrete random variable** if there exists a finite or countably infinite injective sequence $x_1, x_2, \ldots$ such that

- $\mathbb{P}(\xi = x_i) > 0$ for all $i = 1, 2, \ldots;$
- $\mathbb{P}(\xi \in \{x_1, x_2, \ldots\}) = 1.$

Additional notation:

- The sequence $x_1, x_2, \ldots$ is called the set of **values** of ξ, which is denoted by $\mathcal{R}_\xi$.
- The sequence $p_1 := \mathbb{P}(\xi = x_1), p_2 := \mathbb{P}(\xi = x_2), \ldots$ is called the **distribution** of ξ, or simply a **discrete distribution**.

Properties of Distributions of Discrete Random Variables

Theorem

A discrete distribution is a sequence of positive numbers summing to 1, that is $\sum_i p_i = 1$.

Proof. This follows from the σ-additivity of probability:

$$\sum_i p_i = \sum_i \mathbb{P}(\xi = x_i) = \mathbb{P}\left(\bigcup_i (\xi = x_i)\right) = \mathbb{P}\left(\xi \in \{x_1, x_2, \dots\}\right) = 1.$$

$\boxtimes$

Expectation of a Discrete Random Variable

Definition

The **expectation** of a discrete random variable ξ is defined as

$$\mathbb{E}(\xi) := \sum_i x_i p_i,$$

if the series $\sum_i |x_i| p_i$ is convergent (that is the series $\sum_i x_i p_i$ is absolutely convergent).

3. Absolutely Continuous Random Variables

Absolutely Continuous Random Variables

Definition

A random variable ξ is called **absolutely continuous** if there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that

$$\mathbb{F}(x) = \int_{-\infty}^{x} f(t)dt \qquad (x \in \mathbb{R}).$$

Such a function f is called the **density function** of ξ.

Remark

If ξ is absolutely continuous with distribution function $\mathbb{F}$ and density function f, then

$$f(x) = \mathbb{F}'(x)$$

at all points where $\mathbb{F}$ is differentiable.

Properties of the density function

Theorem

Properties of the density function. Let ξ be an absolutely continuous random variable with density function f. Then

1. *f is **nonnegative**, that is, $f(x) \geqslant 0$ for all $x \in \mathbb{R}$. (This is not entirely precise, it can be negative at most on a set of Lebesgue-null measure, but we can require it to be nonnegative, since the value of a density function is allowed to change on a set of Lebesgue-null measure.)*
2. $\int_{-\infty}^{+\infty} f(x)dx = 1$.

Expected Value of an Absolutely Continuous Random Variable

Definition

The **expected value** of an absolutely continuous random variable ξ is defined as

$$\mathbb{E}(\xi) := \int_{-\infty}^{+\infty} x f(x) dx,$$

if the integral $\int_{-\infty}^{+\infty} |x| f(x) dx$ exists and is finite.

The probability of a random variable falling into an interval

Theorem

Let ξ be an random variable and $-\infty \leqslant a < b \leqslant +\infty$.

1. ***The probability of a random variable falling into an interval expressed by distribution function:***

$$\mathbb{P}\left(\xi \in [a, b[\right) = \mathbb{F}(b) - \mathbb{F}(a) \quad \text{és} \quad \mathbb{P}(\xi = x_0) = \mathbb{F}(x_0 + 0) - \mathbb{F}(x_0).$$

2. ***The probability of a discrete random variable falling into an interval:***

$$\mathbb{P}(\xi \in H) = \sum_{x_i \in H} p_i,$$

3. ***The probability of a absolutely continuous random variable falling into an interval:***

$$\mathbb{P}(\xi \in H) = \int_a^b f(x)dx$$

The set H denotes any of the intervals $[a, b]$, $]a, b[$, $[a, b[$, $]a, b]$.

Properties of Expected Value

Theorem

1. ***Transformation formula for random variables.*** *If ξ is a random variable with finite expectec value, $g : \mathbb{R} \to \mathbb{R}$ is measurable (say continuous), and $\mathbb{E}(g(\xi))$ exists, then*
   - *on the **discrete case:*** $\mathbb{E}(g(\xi)) = \sum_i g(x_i) p_i$;
   - *on the **absolutely continuous case:***
     $\mathbb{E}(g(\xi)) = \int_{-\infty}^{+\infty} g(x) f(x) dx$.

Continuation

Theorem

Properties of Expected Value

2. ***Transformation formula for probability vector variables*** *(case $n = 2$). If ξ_1, ξ_2 are random variables with an expected value, $g : \mathbb{R}^2 \to \mathbb{R}$ is a measurable (or say continuous) function such that the random variable of $g(\xi_1, \xi_2)$ also has an expected value, then*
   - ***in the discrete case*** $\mathbb{E}(g(\xi_1, \xi_2)) = \sum_{ij} g(x_i, x_j) p_{ij}$, *where* $p_{ij} := \mathbb{P}(\xi_1 = x_i, \xi_2 = x_j)$ $(i = 1, 2, \dots, j = 1, 2, \dots)$;
   - ***in the absolutely continuous case*** $\mathbb{E}(g(\xi_1, \xi_2)) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x, y) f(x, y) dydx$, *where* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *is a function such that* $\mathbb{P}(\xi_1 < x, \xi_2 < y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u, v) dvdu$. *Similar formulas can be easily formulated for* $n \in \mathbb{Z}_+$, $n \geqslant 2$.

Theorem

Properties of Expected Value

3. ***Additivity of expected value.*** *If* $\xi_1, \xi_2, \ldots, \xi_n$ *are random variables with finite expected value, then*
   - *in case of* $n = 2$: $\mathbb{E}(\xi_1 + \xi_2) = \mathbb{E}(\xi_1) + \mathbb{E}(\xi_2)$;
   - *in case of* $n \geqslant 2$: $\mathbb{E}(\xi_1 + \cdots + \xi_n) = \mathbb{E}(\xi_1) + \cdots + \mathbb{E}(\xi_n)$.
4. ***Homogeneity of expected value.*** *If* ξ *has finite expected value and* $\lambda \in \mathbb{R}$, *then* $\mathbb{E}(\lambda\xi) = \lambda\mathbb{E}(\xi)$.

Continuation

Theorem

Properties of Expectation

5. ***Shift property.*** *If the random variable ξ has the finite expected value and $c \in \mathbb{R}$, then $\mathbb{E}(\xi + c) = \mathbb{E}(\xi) + c$.*
6. ***On the expected value of bounded random variable.*** *If ξ is a random variable such that there exist numbers $-\infty < m < \xi < M < +\infty$, and $\mathbb{P}(m < \xi < M) = 1$, then $m < \mathbb{E}(\xi) < M$.*

4. Moments and Central Moments

Standard deviation and Variance

Definition

If ξ is a random variable for which there exists an expected value $\mathbb{E}(\xi^k)$, then

- $\mathbb{E}(\xi^k)$ is called the k-**th moment** of the random variable ξ,
- $\mathbb{E}\left((\xi - \mathbb{E}(\xi))^k\right)$ is called the **centered moment** of the random variable ξ.
- The second centered moment is called the **variance** and it is denoted by $\mathbb{D}^2(\xi)$.
- If a random variable ξ has a variance, then the number $\mathbb{D}(\xi) = \sqrt{\mathbb{D}^2(\xi)}$ is called the **standard deviation** of the random variable ξ.

Computation of Variance

Theorem

Computation of Variance *If ξ has finite variance, then*

$$\mathbb{D}^2(\xi) = \mathbb{E}(\xi^2) - \mathbb{E}(\xi)^2.$$

Properties of Variance

Theorem

Let ξ, ξ_1, ξ_2 be random variables, such that there exists its variances, and $c \in \mathbb{R}$.

1. ***Quadratically homogeneous***, *that is,* $\mathbb{D}^2(c\xi) = c^2\mathbb{D}^2(\xi)$. *Accordingly, the standard deviation is absolutely homogeneous, that is,* $\mathbb{D}(c\xi) = |c|\mathbb{D}(\xi)$.
2. ***Translation invariant*** $\mathbb{D}^2(\xi + c) = \mathbb{D}^2(\xi)$.
3. ***Positive semidefinite***, *that is,* $\mathbb{D}^2(\xi) \geqslant 0$ *and* $\mathbb{D}^2(\xi) = 0$ *if and only if, there exists* $c \in \mathbb{R}$ *such, that* $\mathbb{P}(\xi = c) = 1$.
4. ***Variance of the sum:*** $\mathbb{D}^2(\xi_1 + \xi_2) = \mathbb{D}^2(\xi_1) + \mathbb{D}^2(\xi_2) + 2\mathrm{cov}(\xi_1, \xi_2)$. *(So additivity property is statisfied in only case, when the random variables are uncorrelated.)*

Example 1

Example

We toss a (fair) die. Let ξ denote the number rolled. Determine the expected value and variance of ξ.

Solution:

The **distribution** of ξ:

x_i	1	2	3	4	5	6
p_i	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

The **expected value** of ξ:

$$\mathbb{E}(\xi) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} =$$
$$= \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = \frac{21}{6} = 3.5$$

Example 1 (continued)

The **second moment** of ξ:

$$\mathbb{E}(\xi^2) = 1^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{6} + 3^2 \cdot \frac{1}{6} + 4^2 \cdot \frac{1}{6} + 5^2 \cdot \frac{1}{6} + 6^2 \cdot \frac{1}{6} =$$
$$= \frac{1}{6}(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2) = \frac{91}{6}$$

The **variance** of ξ:

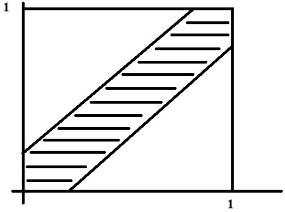
$$\mathbb{D}^2(\xi) = \mathbb{E}(\xi^2) - (\mathbb{E}(\xi))^2 = \frac{91}{6} - 3.5^2 = \frac{35}{12}$$

Example 2

Example

We randomly drop two points, independently of each other, onto the interval $[0,1]$. Let ξ denote the distance between the two points. Determine the distribution function, density function, expected value and variance of ξ.

Solution: This task is similar to the problem described in the section on the geometrical method of probability calculation, when two people arrive randomly and independently within a given time interval at a given place... Here we should think of a similar figure.



Let $x \in [0,1]$. Then

$$\mathbb{P}(\xi < x) = \frac{1 - 2\frac{(1-x)^2}{2}}{1} = 1 - (1-x)^2 = -x^2 + 2x,$$

so the **distribution function** of ξ is:

$$\mathbb{F}(x) = \begin{cases} 0, & \text{if } x < 0, \\ -x^2 + 2x, & \text{if } x \in [0,1], \\ 1, & \text{if } x > 1. \end{cases}$$

The **density function** of ξ is:

$$f(x) = \begin{cases} 0, & \text{if } x \notin [0,1], \\ -2x + 2, & \text{otherwise.} \end{cases}$$

The **expected value** of ξ:

$$\mathbb{E}(\xi) = \int_0^1 x(-2x+2)dx =$$
$$= \int_0^1 (-2x^2+2x)dx = \left[\frac{-2x^3}{3} + \frac{2x^2}{2}\right]_{x=0}^{x=1} = -\frac{2}{3} + 1 = \frac{1}{3}.$$

The **second moment** of ξ:

$$\mathbb{E}(\xi^2) = \int_0^1 x^2(-2x+2)dx =$$
$$= \int_0^1 (-2x^3+2x^2)dx = \left[\frac{-2x^4}{4} + \frac{2x^3}{3}\right]_{x=0}^{x=1} = -\frac{1}{2} + \frac{2}{3} = \frac{1}{6}.$$

The **variance** of ξ:

$$\mathbb{D}^2(\xi) = \mathbb{E}(\xi^2) - [\mathbb{E}(\xi)]^2 = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{18}.$$

End of Lecture 4