Probability Theory and Mathematical Statistics

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Lecture 3

Conditional Probability, Bayes' Theorem, Independence of Events

1. Conditional Probability

Definition of Conditional Probability

Definition

If A, $B \in \mathcal{F}$ are events with $\mathbb{P}(B) \neq 0$, then

$$\mathbb{P}(A|B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

is called the **conditional probability** of event A given B.

Properties of Conditional Probability

Theorem

Let $B \in \mathcal{F}$ with $\mathbb{P}(B) > 0$, and A_1, A_2 events of positive probability.

1. ***Normalization:*** $\mathbb{P}(\Omega|B) = 1$ *(in particular* $\mathbb{P}(B|B) = 1$*);*
2. ***Finite additivity:*** *If $A_1 \cap A_2 = \emptyset$, then*

$$\mathbb{P}(A_1 \cup A_2|B) = \mathbb{P}(A_1|B) + \mathbb{P}(A_2|B);$$

3. ***Complementary event:*** *For any event A,*

$$\mathbb{P}(\overline{A}|B) = 1 - \mathbb{P}(A|B).$$

2. Bayes' Theorem

Bayes' Formula

Theorem

Bayes' Formula *If A, B are events of positive probability, then*

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A|B) \cdot \mathbb{P}(B)}{\mathbb{P}(A)}.$$

Partition of the Sample Space

Definition

The events $B_1, B_2, \ldots, B_n$ form a **partition** of Ω if they are pairwise disjoint and cover Ω, that is

$$B_j \cap B_k = \emptyset \quad (j \neq k), \qquad B_1 \cup B_2 \cup \cdots \cup B_n = \Omega.$$

Similarly, a countable infinite partition can also be defined.

The Law of the Total Probability

Theorem

Law of Total Probability *If $B_1, B_2, \ldots, B_n$ is a partition of Ω consisting of events of positive probability, and A is an event, then*

$$\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A|B_i)\mathbb{P}(B_i).$$

The countable infinite version of the above Law can be formulated similarly.

Bayes' Theorem

Theorem

Bayes' Theorem *If $B_1, B_2, \ldots, B_n$ form a partition of Ω consisting of events of positive probability, and A is an event of positive probability, then*

$$\mathbb{P}(B_j|A) = \frac{\mathbb{P}(A|B_j)\mathbb{P}(B_j)}{\sum_{i=1}^{n} \mathbb{P}(A|B_i)\mathbb{P}(B_i)}.$$

Example

Example

An urn contains 5 balls: 3 red and 2 white. We draw one ball and replace it. If the drawn ball is red, we add 2 more red balls to the urn; if it is white, we add 5 more white balls. Then we draw again. What is the probability that the second ball drawn is red?

Solution

Let

- B_1 be the event that the first drawn ball is red,
- B_2 the event that the first drawn ball is white, and
- A the event that the second drawn ball is red.

Then

$$\mathbb{P}(B_1) = \frac{3}{5}, \quad \mathbb{P}(B_2) = \frac{2}{5}, \quad \mathbb{P}(A|B_1) = \frac{5}{7}, \quad \mathbb{P}(A|B_2) = \frac{3}{10}.$$

By the law of total probability:

$$\begin{aligned} \mathbb{P}(A) = \mathbb{P}(A|B_1)\mathbb{P}(B_1) + \mathbb{P}(A|B_2)\mathbb{P}(B_2) = \\ = \frac{5}{7} \cdot \frac{3}{5} + \frac{3}{10} \cdot \frac{2}{5} = \frac{96}{175} = 0.5486. \end{aligned}$$

The Chain Rule

Theorem

Chain Rule *Let* $B_1, B_2, \ldots, B_n$ *be events such that* $\mathbb{P}(B_1 \cap B_2 \cap \cdots \cap B_{n-1}) \neq 0$. *Then*

$$\mathbb{P}(B_1 \cap B_2 \cap \cdots \cap B_n) = \mathbb{P}(B_1)\mathbb{P}(B_2|B_1) \cdots \mathbb{P}(B_n|B_1 \cap B_2 \cap \cdots \cap B_{n-1}).$$

Motivation for the concept of independence of two events

We think that the events A, and B are independent, if

$$\mathbb{P}(A|B) = \mathbb{P}(A).$$

Problems:

- The above expression requires that $\mathbb{P}(B) \neq 0$;
- and this expression is not symmetrical.

From this eypression by a little calculation we obtain that

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B),$$

and this eypression does not require that $\mathbb{P}(B) \neq 0$, and it is symmetric.

3. Independence of Events

Independence of Two Events

Definition

Let A and B be events. We say that A **and** B **are independent** if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B).$$

Pairwise and Mutual Independence of Finitely Many Events

Definition

We say that the events $A_1, A_2, \ldots, A_n$ are

- **pairwise independent** if for all indices $1 \leqslant i < j \leqslant n$,

$$\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i)\mathbb{P}(A_j);$$

- **mutually independent** if for all integers $2 \leqslant k \leqslant n$ and for all indices $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$,

$$\mathbb{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = \mathbb{P}(A_{i_1})\mathbb{P}(A_{i_2}) \ldots \mathbb{P}(A_{i_k}).$$

Pairwise and Mutual Independence of an Infinite Sequence

Definition

We say that the sequence of events $A_1, A_2, \ldots$ is

- **pairwise independent** if for all indices $i < j$,

$$\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i)\mathbb{P}(A_j);$$

- **mutually independent** if for all integers $2 \leqslant k$ and for all indices $1 \leqslant i_1 < i_2 < \cdots < i_k$,

$$\mathbb{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = \mathbb{P}(A_{i_1})\mathbb{P}(A_{i_2}) \ldots \mathbb{P}(A_{i_k}).$$

Theorem

Theorem

If $A_1, A_2, \ldots$ is a sequence of mutually independent events and any of them (possibly all) are replaced with their complements, then the resulting sequence is still mutually independent.

End of Lecture 3