Probability Theory and Mathematical Statistics

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Lecture 2

Drawing balls from an urn, Geometric method of calculating probability, Kolmogorov probability space

1. Drawing balls from an urn

Notations

We draw balls one by one from an urn. In the case **without replacement**, a drawn ball is not returned to the urn; in the case **with replacement**, it is returned.
In both cases, we use the following notations:

- N is the number of balls in the urn;
- s is the number of red balls;
- $N - s$ is the number of white balls;
- n is the number of balls drawn;
- k denotes the number of red balls drawn;
- p_k denotes the probability that the number of red balls drawn is k.

The set of balls in the urn is denoted by

$$G := \{P_1, P_2, \dots, P_s, F_1, F_2, \dots, F_{N-s}\}.$$

Drawing without replacement

First we construct the set Ω. Since in this case the drawn balls are not returned to the urn, we have

$$\Omega := \{(g_1, g_2, \ldots, g_n) \in G^n \mid \; g_j \neq g_k, \; \text{if } j \neq k\}.$$

Note that the value of k can not be arbitrary, it have to staisfy the following inequality

$$\max(0, n - (N - s)) \leqslant k \leqslant \min(s, n).$$

It is also worth mentioning that

$$\#\Omega = V_N^n = (N)_n.$$

Number of favorable outcomes

Suppose we have drawn n balls and among the $1, 2, \ldots, n$ positions, k positions contain red balls. The positions of the red balls can be chosen in

$$C_n^k = \binom{n}{k}$$

ways.

Once the red and white positions are fixed, these positions can be filled with the appropriate colored balls in

$$V_s^k \cdot V_{N-s}^{n-k} = (s)_k (N-s)_{n-k}$$

ways.

Number of favorable and total outcomes

Number of favorable outcomes:

$$\binom{n}{k}(s)_k(N-s)_{n-k}.$$

Total number of outcomes:

$$V_N^n = (N)_n.$$

Probability of drawing without replacement

The probability:

$$p_k = \mathbb{P}(A_k) = \frac{\binom{n}{k}(s)_k(N-s)_{n-k}}{(N)_n} =$$

$$= \frac{\frac{n!}{(n-k)!k!}(s)_k(N-s)_{n-k}}{(N)_n} = \frac{\frac{(s)_k}{k!}\,\frac{(N-s)_{n-k}}{(n-k)!}}{\frac{(N)_n}{n!}} = \frac{\binom{n}{k}\binom{N-s}{n-k}}{\binom{N}{n}}.$$

Drawing with replacement

First we construct the set Ω. Since the balls are returned to the urn after each draw, we have

$$\Omega := G \times G \times \cdots \times G = G^n.$$

Analogously to the case without replacement, we obtain:
Number of favorable outcomes:

$$C_n^k \cdot V_s^{k(i)} \cdot V_{N-s}^{n-k(i)} = \binom{n}{k} s^k (N-s)^{n-k}.$$

Total number of outcomes:

$$V_N^{n(i)} = N^n.$$

Probability of drawing with replacement

The desired probability: From the favorable/total formula we get

$$p_k = \mathbb{P}(A_k) = \frac{\binom{n}{k} s^k (N-s)^{n-k}}{N^n} = \binom{n}{k} \frac{s^k}{N^k} \cdot \frac{(N-s)^{n-k}}{N^{n-k}} =$$

$$= \binom{n}{k} \left(\frac{s}{N}\right)^k \left(1 - \frac{s}{N}\right)^{n-k}.$$

2. Geometric method of calculating probability

Geometric probability

Let $\mathcal{K}$ be an experiment. To each possible outcome of the experiment we assign a point in $\mathbb{R}^n$. The resulting set of points is denoted by Ω.

The geometric method of calculating probability for an event $A \subseteq \Omega$ can be applied if there exists a measure m (typically length, area, or volume) such that

$$0 < m(\Omega) < +\infty$$

and the set $A \subseteq \mathbb{R}^n$ is also measurable with respect to m. Then the probability of event A is

$$\mathbb{P}(A) = \frac{m(A)}{m(\Omega)}.$$

Example of geometric probability

Example

Two people arrive randomly at a given place between 12 am and 2 pm and each stays for 20 minutes. What is the probability that they meet?

Solution:

$$\Omega := [0,120] \times [0,120], \qquad A := \{(x,y) \in \Omega \mid \ |x-y| < 20\}.$$

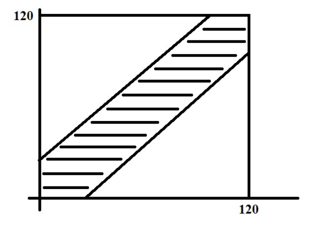
It is easy to see that

$$|x-y| < 20 \quad \iff \quad y > x - 20 \text{ and } y < x + 20.$$

Continuation

Example

The inequalities define two half-planes; intersecting them with Ω gives the shaded strip in the figure below.



The required probability is

$$\mathbb{P}(A) = \frac{m(A)}{m(\Omega)} = \frac{120^2 - 2 \cdot \frac{100^2}{2}}{120^2} = 1 - \left(\frac{100}{120}\right)^2 = \frac{11}{36} = 0.3056.$$

3. Kolmogorov probability space

2.3.1. The concept and properties of a σ-algebra

The sort history of Probability Theory

The classical probability space were first studied by Pascal and Fermat.

Blase Pascal (1623 – 1662) was a French mathematician, physicist, philosopher, and Catholic writer.

Pierre de Fermat (1601 – 1665) was a French magistrate, polymath, and above all mathematician.

Andrey Nikolaevich Kolmogorov (1903 - 1987) was a Soviet mathematician who placed probability theory on a measure-theoretic basis. It is mentioning the Kolmogorov's article in 1988.

σ-algebra

Definition

Let X be a non-empty set, and let $\mathcal{A} \subseteq \mathcal{P}(X)$ be a family of subsets. We say that $\mathcal{A}$ **is a σ-algebra on** X if

1. $X \in \mathcal{A}$;
2. If $A \in \mathcal{A}$, then its complement $\overline{A} := X \setminus A \in \mathcal{A}$;
3. If $A_1, A_2, \ldots$ is a sequence of sets in $\mathcal{A}$, then

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}.$$

Properties of σ-algebras

Theorem

If $\mathcal{A}$ is a σ-algebra on X, then

1. $\emptyset \in \mathcal{A}$;
2. *If (A_n) is a sequence of sets in $\mathcal{A}$, then $\bigcap_{i=1}^{\infty} A_i \in \mathcal{A}$;*
3. *If $A, B \in \mathcal{A}$, then $A \cup B$, $A \cap B$, and $A \setminus B \in \mathcal{A}$;*
4. *If $A_1, A_2, \ldots, A_n$ are finitely many sets in $\mathcal{A}$, then $A_1 \cup A_2 \cup \cdots \cup A_n \in \mathcal{A}$ and $A_1 \cap A_2 \cap \cdots \cap A_n \in \mathcal{A}$.*

(So a σ-algebra contains $\emptyset$, and it is closed under countable set operations.)

3. Kolmogorov probability space

2.3.2. The Kolmogorov probability space

Kolmogorov probability space

Definition

A triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a **Kolmogorov probability space** if

1. Ω is a non-empty set, called the **sample space**, whose elements are the **elementary events**. Elementary events are not events.
2. $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ is a σ-algebra on Ω. $\mathcal{F}$ is the **set of events**.
3. $\mathbb{P} : \mathcal{F} \to \mathbb{R}$ is a function such that:

I. **Non-negative**: $\mathbb{P}(A) \geqslant 0$ for all $A \in \mathcal{F}$;

II. **Normalized to 1**: $\mathbb{P}(\Omega) = 1$;

III. σ-**additive**: if $A_1, A_2, \ldots$ is a sequence of pairwise disjoint events ($A_i \in \mathcal{F}$, $i = 1, 2, \ldots$, and $A_j \cap A_k = \emptyset$ if $j \neq k$), then

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n).$$

Properties of probability 1-3.

Theorem

If $(\Omega, \mathcal{F}, \mathbb{P})$ is a Kolmogorov probability space, then

1. $\mathbb{P}(\emptyset) = 0$.
2. ***The finite additivity of probability***, *that is*

$$\mathbb{P}(A_1 \cup A_2 \cup \cdots \cup A_n) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \cdots + \mathbb{P}(A_n)$$

   for all A_1, A_2, ..., A_n pairwise disjoint events.

3. ***The probability of the complementary event***

$$\mathbb{P}\left(\overline{A}\right) = 1 - \mathbb{P}(A)$$

   for all $A \in \mathcal{F}$.

Properties of probability 4-6.

Theorem

If $(\Omega, \mathcal{F}, \mathbb{P})$ *is a Kolmogorov probability space, then*

4. *If* $A, B \in \mathcal{F}$ *such that* $A \subseteq B$*, then*

$$\mathbb{P}(B \setminus A) = \mathbb{P}(B) - \mathbb{P}(A).$$

5. ***The monotonicity of probability*** *If* $A, B \in \mathcal{F}$ *such that* $A \subseteq B$*, then*

$$\mathbb{P}(A) \leqslant \mathbb{P}(B).$$

6. $0 \leqslant \mathbb{P}(A) \leqslant 1$ *for all* $A \in \mathcal{F}$*.*

Properties of probability 7-9.

7. **Inclusion-exclusion formula for two events:**

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

   for all $A, B \in \mathcal{F}$.

8. **Inclusion-exclusion formula for three events:**

$$\mathbb{P}(A \cup B \cup C) =$$
$$= \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C).$$

   for all $A, B, C \in \mathcal{F}$.

9. **Poincaré formula:**

$$\mathbb{P}(A_1 \cup \cdots \cup A_n) = \sum_{k=1}^{n} \sum_{1 \leqslant i_1 < \cdots < i_k} (-1)^{k+1} \mathbb{P}(A_{i_1} \cap \cdots \cap A_{i_k}).$$

   for all $A_1, A_2, \ldots, A_n$ are events.

Continuity of probability

Theorem

Continuity of probability

1. *If* (B_n) *is a decreasing sequence of events, that is,*

$$B_1 \supseteq B_2 \supseteq \dots$$

and $\bigcap_{n=1}^{\infty} B_n = \emptyset$, *then* $\lim_{n\to\infty} \mathbb{P}(B_n) = 0$.

2. *If* (B_n) *is a decreasing sequence of events with* $\bigcap_{n=1}^{\infty} B_n = B$, *then* $\lim_{n\to\infty} \mathbb{P}(B_n) = \mathbb{P}(B)$.
3. *If* (B_n) *is an increasing sequence of events, that is,*

$$B_1 \subseteq B_2 \subseteq \dots,$$

with $\bigcup_{n=1}^{\infty} B_n = B$, *then* $\lim_{n\to\infty} \mathbb{P}(B_n) = \mathbb{P}(B)$.

End of Lecture 2