Probability and Mathematical Statistics

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Lecture 1

Sets, Combinatorics, Classical probability space

1. Basic mathematical preliminaries

Number sets

We use the following notations:

- **Set of positive integers:** $\mathbb{Z}_+ := \{1, 2, 3, \ldots\}$.
- **Set of integers:** $\mathbb{Z} := \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.
- **Set of rational numbers:** $\mathbb{Q} := \left\{ \frac{p}{q} \,\middle|\, p \in \mathbb{Z}, q \in \mathbb{Z}_+ \right\}$.
- **Set of real numbers:** $\mathbb{R}$ (Think of them as the points on the real line.)
- **Set of complex numbers:** $\mathbb{C} := \{a + bi \mid a, b \in \mathbb{R}\}$, where i is a number such that $i^2 = -1$.

Basic set theory

Definition

If the element a belongs to the set A, then it is denoted by $a \in A$. If the element a does not belong to the set A, then it is denoted by $a \notin A$.

- **Inclusion:** We say that the set A **is a subset of the set** B if every element of A is also an element of B. Notation: $A \subseteq B$.
- **Universal set:** Usually we fix a universal set X, and the sets we consider are subsets of the set X.

Power set

Definition

The set of all subsets of the set X is denoted by $\mathcal{P}(X)$ (Power set of the set X).

Example *Let* $X := \{1, 2, 3\}$. *Then*

$$\mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

The example above suggests the following theorem:

Theorem

The n-element set has 2^n *subsets.*

Set operations

Definition

The most important set operations are intersection, union, difference, and complement. Let A, B be sets, and X be the universal set.

- **Intersection of A and B:** $A \cap B := \{x \in X \mid x \in A \text{ and } x \in B\}$.
- **Union of A and B:** $A \cup B := \{x \in X \mid x \in A \text{ or } x \in B\}$.
- **Difference of A and B:** $A \setminus B := \{x \in X \mid x \in A \text{ and } x \notin B\}$.
- **Complement of A:** $\overline{A} := X \setminus A$.
- **Cartesian product of A and B:** $A \times B := \{(a, b) \mid a \in A \text{ and } b \in B\}$.

Set operations with series of sets

Definition

Let $A_1, A_2, \ldots$ be subsets of the universal set X. Then we can define the intersection and the union of the sets A_i by

$$\bigcap_{i=1}^{\infty} A_i := \{x \in X | \forall i \in \mathbb{Z}_+ : x \in A_i\};$$

$$\bigcup_{i=1}^{\infty} A_i := \{x \in X | \exists i \in \mathbb{Z}_+ : x \in A_i\};$$

where $\forall$ denotes "the all elements...", and $\exists$ denotes "there exists an element...".

De Morgan's laws

Theorem

Let A, B be subsets of the universal set X. Then

$$\overline{A \cap B} = \overline{A} \cup \overline{B} \qquad \overline{A \cup B} = \overline{A} \cap \overline{B}.$$

Theorem

Let $A_1, A_2, \ldots$ be a sequence of subsets of the universal set X. Then

$$\overline{\bigcap_{i=1}^{\infty} A_i} = \bigcup_{i=1}^{\infty} \overline{A_i}, \qquad \overline{\bigcup_{i=1}^{\infty} A_i} = \bigcap_{i=1}^{\infty} \overline{A_i}.$$

Finite and infinite sets

Definition

A set X is said to be **finite**, if

- it has 0 element, that is ($X = \emptyset$);
- it has 1 element, that is (X is a singleton);
- it has 2 elements; $\dots$

A set X is said to be **infinite**, if X is not finite.

Definition

The number of the elements of a finite sets X is said to be **cardinality** of the set X, and it is denoted by $\#X$.

Definition

An X infinite set is said to be **countable infinite**, if it can be placed in one-to-one correspondence with the set of possitive integers $\{1, 2, 3, \dots\}$.

On the cardinality of finite sets

Theorem

If A and B are finite sets, then

- $\#(A \times B) = \#A \cdot \#B$;
- *morever, if A, and B are disjoint sets (that is $A \cap B = \emptyset$), then*

$$\#(A \cup B) = \#A + \#B.$$

2. Combinatorics

Mathematical tools

Factorial function:

$$n! := \begin{cases} 1, & \text{if } n = 0; \\ 1 \cdot 2 \cdots n & , \text{ if } n \in \mathbb{Z}_+. \end{cases}$$

Binomial coefficient:

$$\binom{n}{k} := \begin{cases} \frac{n!}{(n-k)! \cdot k!} = \frac{n(n-1)\cdots(n-k+1)}{1 \cdot 2 \cdots k} = \frac{(n)_k}{k!} & , \text{ if } 0 \leqslant k \leqslant n; \\ 0 & , \text{ otherwise.} \end{cases}$$

Multinomial coefficient:

$$\binom{n}{k_1 k_2 \ldots k_r} := \begin{cases} \frac{n!}{k_1! \cdot k_2! \cdots k_r!}, & \text{if } 0 \leqslant k_i \leqslant n \text{ and } k_1 + \cdots + k_r = n; \\ 0 & , \text{ otherwise.} \end{cases}$$

Permutations

Definition

1. **Permutation without repetition** — we seek all possible orders of n distinct elements. The number of such sequences are denoted by P_n.
2. **Permutation with repetition** — among the n elements, k_1, $k_2, \dots, k_r$, $k_1 + k_2 + \cdots + k_r = n$ have the same properties, where the order among elements with the same properties does not matter, then the number of such sequences is denoted by $P_n^{k_1,\dots,k_r(r)}$.

Theorem

Number of permutations without and with repetition

$$P_n = n! \qquad (n \in \mathbb{Z}_+),$$

$$P_n^{k_1,k_2,\dots,k_r(r)} = \binom{n}{k_1 k_2 \dots k_r} := \frac{n!}{k_1! k_2! \cdots k_r!}$$

Example of Permutation without repetition

How many different orders are the letters A, B, C?

Example of Permutation without repetition

How many different orders are the letters A, B, C?
The orders are:
ABC, ACB, BAC, BCA, CAB, CBA.
There are 6 different orders.

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By formula:

$$P_3 = 3! = 3 \cdot 2 \cdot 1 = 6.$$

Example of permutation with repetition

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$AABB$, $ABAB$, $ABBA$, $BAAB$, $BABA$, $BBAA$.
There are 6 different orders.

Example of permutation with repetition

How many different orders are the letters A, A, B, B?
The orders are:
$AABB$, $ABAB$, $ABBA$, $BAAB$, $BABA$, $BBAA$.
There are 6 different orders.
By formula:

$$P_4^{2,2(r)} = \frac{4!}{2! \cdot 2!} = \frac{24}{4} = 6.$$

Variations

Definition

The k-element variation of n elements means that we choose k items from n elements such that the order of the chosen elements matters.

1. **Variations without repetition** — each element can be chosen at most once. The number of k-element variations without repetition from n elements is denoted by V_n^k.
2. **Variations with repetition** — an element can be chosen more than once. The number of k-element variations with repetition from n elements is denoted by $V_n^{k(r)}$.

Theorem

Number of k-element variations without and with repetition from n elements

$$V_n^k = n(n-1)\cdots(n-k+1) := (n)_k, \qquad V_n^{k(r)} = n^k.$$

Example of Variation without repetition

How many different 3-letter sequences can be formed from the letters A, B, C, D, E, F if each letter can only appear once?

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The orders are:

ABC, ABD, ABE, ABF, ACB, ACD, ACE, ACF, ADB, ADC, ADE, ADF, $\ldots$

There are 120 different orders.

Example of Variation without repetition

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The orders are:

ABC, ABD, ABE, ABF, ACB, ACD, ACE, ACF, ADB, ADC, ADE, ADF, $\ldots$

There are 120 different orders.

By formula:

$$V_6^3 = 6 \cdot 5 \cdot 4 = 120.$$

Example of Variation with repetition

How many different 3-letter sequences can be formed from the letters A and B, if each letter can appear multiple times?

Example of Variation with repetition

How many different 3-letter sequences can be formed from the letters A and B, if each letter can appear multiple times?

The orders are:

AAA, AAB, AAC, AAD, AAE, AAF, ABA, ACA, ...

There are 216 different orders.

Example of Variation with repetition

How many different 3-letter sequences can be formed from the letters A and B, if each letter can appear multiple times?

The orders are:

$AAA, AAB, AAC, AAD, AAE, AAF, ABA, ACA, \ldots$

There are 216 different orders.

By formula:

$$V_6^{3(r)} = 6 \cdot 6 \cdot 6 = 216.$$

Combinations

Definition

We choose k elements from n elements such that the order of the chosen elements does not matter.

1. **Combinations without repetition** — each element can be chosen at most once. The number of k-element combinations without repetition from n elements is denoted by C_n^k.
2. **Combinations with repetition** — an element can be chosen more than once. The number of k-element combinations with repetition from n elements is denoted by $C_n^{k(r)}$.

Combinations

Theorem

Number of k-element combinations without and with repetition from n elements

$$C_n^k = \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{1\cdot 2\cdots k} = \frac{(n)_k}{k!},$$

$$C_n^{k(r)} = \binom{n+k-1}{k} = \frac{(n+k-1)_k}{k!}.$$

$\binom{n}{k}$ *is read as "n choose k".*

Example of Combination without repetition

How many subsets does set $A = \{1, 2, 3, 4, 5\}$ have?

Example of Combination without repetition

How many subsets does set $A = \{1, 2, 3, 4, 5\}$ have?

The subsets are:

0-element subsets: $\emptyset$

1-element subsets: $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$,

2-element subsets: $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{1, 5\}$, $\{2, 3\}$, $\{2, 4\}$, $\{2, 5\}$, $\{3, 4\}$, $\{3, 5\}$, $\{4, 5\}$,

3-element subsets: $\{1, 2, 3\}$, $\{1, 2, 4\}$ $\ldots$,

4-element subsets: $\{1, 2, 3, 4\}$, $\{1, 2, 3, 5\}$, $\{1, 2, 4, 5\}$, $\{1, 3, 4, 5\}$, $\{2, 3, 4, 5\}$,

5-element subsets: $\{1, 2, 3, 4, 5\}$.

There are 32 different subsets.

Example of Combination without repetition

How many subsets does set $A = \{1, 2, 3, 4, 5\}$ have?

The subsets are:

0-element subsets: $\emptyset$

1-element subsets: $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$,

2-element subsets: $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{1, 5\}$, $\{2, 3\}$, $\{2, 4\}$, $\{2, 5\}$, $\{3, 4\}$, $\{3, 5\}$, $\{4, 5\}$,

3-element subsets: $\{1, 2, 3\}$, $\{1, 2, 4\}$ $\ldots$,

4-element subsets: $\{1, 2, 3, 4\}$, $\{1, 2, 3, 5\}$, $\{1, 2, 4, 5\}$, $\{1, 3, 4, 5\}$, $\{2, 3, 4, 5\}$,

5-element subsets: $\{1, 2, 3, 4, 5\}$.

There are 32 different subsets.
By formula:

$$C_5^0 + C_5^1 + C_5^2 + C_5^3 + C_5^4 + C_5^5 = \binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5} =$$

$$= 1 + 5 + 10 + 10 + 5 + 1 = 32$$

Example of Combination with repetition

There are 5 flavors of ice cream. How many ways can we get 3 scoops of ice cream, not counting the order of the scoops?

Example of Combination with repetition

There are 5 flavors of ice cream. How many ways can we get 3 scoops of ice cream, not counting the order of the scoops?

By formula:

$$C_5^{3(r)} = \binom{5+3-1}{3} = \binom{7}{3} = 35.$$

Properties of binomial coefficients

Theorem

1. $\binom{n}{0} = \binom{n}{n} = 1;$
2. $\binom{n}{k} = \binom{n}{n-k};$
3. $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$

Pascal's triangle

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

$$1$$
$$1 \quad 1$$
$$1 \quad 2 \quad 1$$
$$1 \quad 3 \quad 3 \quad 1$$
$$1 \quad 4 \quad 6 \quad 4 \quad 1$$
$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$

The binomial and multinomial theorems

Theorem

1. ***Binomial theorem:***

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

for all $n \in \mathbb{Z}_+ \cup \{0\}$, $a, b \in \mathbb{R}$;

2. ***Multinomial theorem:***

$$(a_1 + a_2 + \cdots + a_r)^n = \sum_{\substack{k_1+k_2+\cdots+k_r=n \\ k_i \geqslant 0}} \binom{n}{k_1\, k_2 \ldots k_r} a_1^{k_1} a_2^{k_2} \ldots a_r^{k_r}$$

for all $n \in \mathbb{Z}_+ \cup \{0\}$, $r \in \mathbb{Z}_+$, *and* $a_1, a_2, \ldots, a_r \in \mathbb{R}$.

3. The classical probability space

Classical probability space

Definition

A triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a **classical probability space** if:

- Ω is a non-empty finite set. It is called the **sample space**, elements of Ω are the **elementary events**. (Elementary events are not events.)
- $\mathcal{F} = \mathcal{P}(\Omega)$ is the **set of events**.
- $\mathbb{P} : \mathcal{F} \to \mathbb{R}$ is the **probability function**, defined by

  $$\mathbb{P}(A) := \frac{\#A}{\#\Omega} \qquad (A \in \mathcal{P}(\Omega)),$$

  where $\#A$ and $\#\Omega$ denote the cardinality of A and Ω, respectively.

Example 1 of a classical probability space

We toss a die. In the sequel a die is always going to be standard. What is the probability of tossing an odd number?

Solution

This experiment is described by a classical probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\Omega = \{1, 2, 3, 4, 5, 6\}$.
Then $A = \{1, 3, 5\}$ is the event in question.
The probability of the event A is

$$\mathbb{P}(A) = \frac{\#A}{\#\Omega} = \frac{3}{6} = \frac{1}{2}.$$

Example 2 of a classical probability space

We toss a die three times. What is the probability that at least two number 6 occur among the tosses?

Solution

This experiment is described by a classical probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where

$$\Omega = \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\},$$

so the total number of outcomes is 6^3.
The favorable outcomes are

$$A := \{(6,6,1),\dots,(6,6,5),(6,1,6),\dots \\ \dots,(6,1,5),(1,6,6),\dots,(5,6,6),(6,6,6)\}$$

The number of favorable outcomes is $\#A := 3 \cdot 5 + 1 = 16$, thus we get that:

$$\mathbb{P}(A) = \frac{16}{6^3} = \frac{2}{27} = 0.0740.$$

Properties of the classical probability space

Theorem

If the triple $(\Omega, \mathcal{F}, \mathbb{P})$ is a classical probability space, then the function $\mathbb{P} : \mathcal{F} \to \mathbb{R}$ is:

- ***Non-negative***: $\mathbb{P}(A) \geqslant 0$ *for all* $A \subseteq \Omega$;
- ***Normalized to 1***: $\mathbb{P}(\Omega) = 1$;
- ***Finitely additive***: *if* $A_1, A_2, \ldots, A_k$ *are pairwise disjoint events, then*

$$\mathbb{P}(A_1 \cup A_2 \cup \cdots \cup A_k) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \cdots + \mathbb{P}(A_k).$$

End of Lecture 1