University of Miskolc

The Faculty of Mechanical Engineering and Information Science

The role of the maximum operator in the theory of measurability and some applications

PhD Thesis

by

Nutefe Kwami Agbeko

The József Hatvany Doctoral School for
Information Science, Engineering and Technology

Head of the PhD School
Prof. Dr. Tibor Tóth
Doctor of the Hungarian Academy of Sciences

Supervisor
Gabriella Vadászné Bognár dr. habil
CSc in Math.

Miskolc, 2009

Jury

Chairman:

Dr. Jenő Szigeti CSc. habil full professor (ME)

Secretary:

Dr. Imre Juhász CSc. associate professor (ME)

Members:

Dr. László Kovács PhD associate professor (ME)

Dr. János Sztrik MTA doctor, associate professor (DE)

Dr. Mihály Hujter CSc. associate professor (BME)

Alternate member:

Dr. Kálmán Liptai associate professor (EKF)

Reviewers:

Dr. Miklós Laczkovich DSc. full professor (ELTE)

Dr. Miklós Rontó DSc. full professor (ME)

Contents

1 Introduction and some notations

The present thesis aims to conduct some studies of the maximum (supremum) operator on:

i.) σ-algebras, i.e. to define on σ-algebras functions, called optimal measures, which map finite union into the maximum under certain restrictions,

ii.) the set of measurable functions with various characterizations.

The material is presented essentially in seven chapters almost all of which begin with an introductory part. In the first some historical backgrounds are presented. Chapters *II–V* deal essentially with results in connection with optimal measure which is a function, continuous from above and suitably normalized, mapping any given σ-algebra into the unit interval $[0,\, 1]$ such that every finite union is mapped into the maximum of the maps of the respective terms. We point out that the choice of the term *optimal measure* is deliberate, since taking the maximum also encounters the meaning given in the *Oxford Dictionary* to the world "optimal". Chapter VI treats some maximal inequalities regarding random variables. In the last chapter we present some informatics simulations.

1.) $\bigvee$ and $\vee$ (respectively, $\bigwedge$ and $\wedge$) stand for the maximum or supremum (respectively the minimum or infimum) operators.

2.) $(\Omega,\, \mathcal{F})$ will denote an arbitrary measurable space, to be specified in especial cases.

2 Optimal measure and the structure theorem

Thesis 1.

In the image of the σ-additive measure (or probability measure) we propose a set function, called optimal measure, which maps σ-algebras into the unit interval $[0,\, 1]$. We showed that every optimal measure is entirely generated by a countable set of the so-called indecomposable atoms.

2.1 Definition, some properties and examples

Definition 2.1 (Agbeko, [5]) *A set function $p : \mathcal{F} \to [0,\, 1]$ will be called optimal measure if it satisfies the following three axioms:*

Axiom 1. *The identities $p(\Omega) = 1$ and $p(\varnothing) = 0$ hold.*

Axiom 2. *For all measurable sets B and E, we have $p(B \cup E) = p(B) \vee p(E)$.*

Axiom 3. *Function p is continuous from above, i.e. whenever $(E_n) \subset \mathcal{F}$ is a decreasing sequence, then $p\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} p(E_n) = \bigwedge_{n=1}^{\infty} p(E_n)$,* The triple $(\Omega, \mathcal{F}, p)$ is referred to as an *optimal measure space.*

An important property can be stated as follows:

Lemma 2.1 (Agbeko, [5]) *Every optimal measure is continuous from below, i.e. whenever* $(B_n) \subset \mathcal{F}$ *is an arbitrary sequence tending increasingly to a measurable set* B*, and* p *an optimal measure, then* $\lim_{n\to\infty} p(B_n) = p(B)$.

I would like to note that the following example was proposed by Prof. M. Laczkovich, to replace the imperfect one I gave in an earlier version.

Example 2.1 (Agbeko, [5]) *Let* $(\Omega, \mathcal{F})$ *be a measurable space,* $(\omega_n) \subset \Omega$ *be a fixed sequence, and* $(\alpha_n) \subset [0, 1]$ *a given sequence tending decreasingly to zero. The function* $p : \mathcal{F} \to [0, 1]$*, defined by*

$$p(B) = \max\{\alpha_n : \omega_n \in B\} \tag{1}$$

is an optimal measure.
Moreover, if $\Omega = [0, 1]$ *and* $\mathcal{F}$ *is a* σ*-algebra of* $[0, 1]$ *containing the Borel sets, then every optimal measure defined on* $\mathcal{F}$ *can be obtained as in* (1).

2.2 The structure theorem

Definition 2.2 (Agbeko, [6]) *By a* p*-atom we mean a measurable set* $H, p(H) > 0$ *such that whenever* $B \in \mathcal{F}$ *and* $B \subset H$*, then* $p(B) = p(H)$ *or* $p(B) = 0$.

Definition 2.3 (Agbeko, [6]) *A* p*-atom* H *is decomposable if there exists a subatom* $B \subset H$ *such that* $p(B) = p(H) = p(H \backslash B)$*. If no such subatom exists, we shall say that* H *is indecomposable.*

The Structure Theorem (Agbeko, [6]) Let $(\Omega, \mathcal{F}, p)$ be an optimal measure space. Then there exists a collection $\mathcal{H}(p) = \{H_n : n \in J\}$ of disjoint indecomposable p-atoms, where J is some countable (i.e. finite or countably infinite) index set, such that for every measurable set $B \in \mathcal{F}$ with $p(B) > 0$ we have

$$p(B) = \max\{p(B \cap H_n) : n \in J\}. \tag{2}$$

Moreover, if J is countably infinite, then the only limit point of the set $\{p(H_n) : n \in J\}$ is 0.

3 Optimal average

Thesis 2.

In the image of the Lebesgue integral (or mathematical expectation), we defined a non-linear functional (called optimal average) for non-negative measurable simple functions and then extend it to arbitrary non-negative measurable functions. Optimal average provides us with many similar well-known results in measure theory, the Fubini and Radon-Nikodym theorems, say.

Let $s = \sum_{i=1}^{n} b_i \chi(B_i)$ be an arbitrary non-negative measurable simple function, where $\{B_i : i = 1, \ldots, n\} \subset \mathcal{F}$ is a partition of Ω.

Lebesgue integral of s :	Optimal average of s : (Agbeko, [5])
$\int_\Omega s d\mu := \sum_{k=1}^{n} b_k \mu(B_k)$	$\diagdown\!\!\!\int_\Omega s dp := \bigvee_{i=1}^{n} b_i p(B_i)$,

It is well-known that in general a measurable simple function can have many decompositions. The question thus arises whether or not the optimal average depends on the decomposition of the simple function. The following result gives a satisfactory answer to this question.

Theorem 3.1 (Agbeko, [5]) *Let $\sum_{i=1}^{n} b_i \chi(B_i)$ and $\sum_{k=1}^{m} c_k \chi(C_k)$ be two decompositions of a measurable simple function $s \geq 0$, where $\{B_i : i = 1, \ldots, n\}$ and $\{C_k : k = 1, \ldots, m\} \subset \mathcal{F}$ are partitions of Ω. Then*

$$\bigvee_{i=1}^{n} b_i p(B_i) = \bigvee_{k=1}^{m} c_k p(C_k).$$

Proposition 3.1 *Let $f \geq 0$ be any bounded measurable function. Then*

$$\sup_{s \leq f} \diagdown\!\!\!\int_\Omega s dp = \inf_{\overline{s} \geq f} \diagdown\!\!\!\int_\Omega \overline{s} dp,$$

where s and $\overline{s}$ denote non-negative measurable simple functions.

Definition 3.1 *The optimal average of a measurable function f is defined by $\diagdown\!\!\!\int_\Omega |f| dp = \sup \diagdown\!\!\!\int_\Omega s dp$, where the supremum is taken over all measurable simple functions $s \geq 0$ for which $s \leq |f|$.*

Remark 3.1 (Agbeko, [6]) *If a function $f : \Omega \to \mathbb{R}$ is measurable, then it is constant almost everywhere on every indecomposable atom.*

Proposition 3.2 (Agbeko, [6]) *Let $p \in \mathcal{P}$ and f be any measurable function. Then*

$$\int_{\Omega} |f|\, dp = \sup\left\{ \int_{H_n} |f|\, dp : n \in J \right\},$$

where $\mathcal{H}(p) = \{H_n : n \in J\}$ is a p-generating countable system.
Moreover if $\int_{\Omega} |f|\, dp < \infty$, then $\int_{\Omega} |f|\, dp = \sup\{c_n \cdot p(H_n) : n \in J\}$, *where $c_n = f(\omega)$ for almost all $\omega \in H_n$, $n \in J$.*

4 Some convergence theorems related to measurable functions

Thesis 3.

By means of optimal measures and averages we were able to characterize various notions of well-known convergence such as the notions of discrete, equally, uniform and pointwise convergence of sequences of measurable functions. The boundedness of sequences of measurable functions were also characterized using the same tools.

Definition 4.1 (Á. Császár and M. Laczkovich, [16, 16, 18]) *LegyenLet X be an arbitrary nonempty set. We say that a sequence of real-valued functions (h_n) converges to a real-valued function h:*
(i) discretely if for every $x \in X$ there exists a positive integer $n_0(x)$ such that $h_n(x) = h(x)$, whenever $n > n_0(x)$;
(ii) equally if there is a sequence (b_n) of positive numbers tending to 0 and for every $x \in X$ there can be found an index $n_0(x)$ such that $|h_n(x) - h(x)| < b_n$ whenever $n > n_0(x)$.

Theorem 4.1 (Agbeko, [7]) *Let (f_n) be any sequence of measurable functions. Then (f_n) tends to a measurable function f pointwise if and only if (z_n) tends to 0 pointwise on $\mathcal{P}_{<\infty}$, where for every $n \in \mathbb{N}$, z_n is defined on $\mathcal{P}_{<\infty}$ by $z_n(p) = \int_{\Omega} |f_n - f|\, dp$.*

Theorem 4.2 (Agbeko, [7]) *A sequence of measurable functions (f_n) converges to some measurable function f equally (resp. discretely) if and only if sequence (z_n) converges to 0 equally (resp. discretely) on $\mathcal{P}_{<\infty}$, where for every $n \in \mathbb{N}$, z_n is defined on $\mathcal{P}_{<\infty}$ by $z_n(p) = \int_{\Omega} |f_n - f|\, dp$.*

5 Maximal inequalities related with probability theory

Thesis 4.

Maximal inequalities in connection with concave (convex) Young functions are discussed. Further we isolated a subset $\mathfrak{A}$ of the set $\mathcal{Y}_{\mathrm{conc}}$ of concave Young functions and showed that it is closed under the composition operation. We also demonstrated that subset $\mathfrak{A}$ is a dense set in $\mathcal{Y}_{\mathrm{conc}}$ with respect to a specific metric and characterized the set of those concave Young functions possessing a positive fixed point.

Definition 5.1 *A function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a* concave Young function *if for all $x \geq 0$ it is defined by*

$$\Phi(x) = \int_0^x \varphi(t)\, dt,$$

where $\Phi(0) = 0$ and $\varphi : (0, \infty) \to (0, \infty)$ is a decreasing right-continuous such that φ is integrable on every finite interval $(0, x)$. All along we assume that $\Phi(\infty) = \infty$.

The set of all concave Young functions will be denoted by $\mathcal{Y}_{\mathrm{conc}}$.

5.1 Maximal inequalities for non-negative submartingales related with concave Young-functions

Definition 5.2 *We say that for the concave Young function Φ the maximal inequality is valid with some positive constant K_Φ (depending only on Φ) if for an arbitrary non-negative submartingale $(X_n, \mathcal{F}_n)$, $n \in \mathbb{N}$, the inequality*

$$E\Phi(X_n^*) \leq K_\Phi (1 + EX_n) \tag{3}$$

holds for all $n \in \mathbb{N}$, with $X_n^* = \bigvee_{k=1}^{n} X_k$.

Theorem 5.1 (Agbeko, [3]) Let Φ be any concave Young function. In order that Φ satisfy the above maximal inequality, it is necessary and sufficient that

$$A_\Phi := \int_1^\infty \frac{\varphi(t)}{t} dt < \infty. \tag{4}$$

Moreover, if $A_\Phi < \infty$, then $K_\Phi = \max(\Phi(1), A_\Phi)$.

5.2 The fixed points of a class of concave Young-functions

In dynamic models, stationary equilibrium is typically described as a solution of the equation $x = f(x)$, where f is a mapping which determines the current state as a function of the previous state, or as a function of the expected future state. In many cases x is a finite dimensional vector, and in general positive solutions (i.e. fixed points of f) are rather sought for.

I should mention that there are different types of Fixed Point theorems. Perhaps the most widely investigated is the one in connection with contractive mappings. Here I would like to mention the name of my colleague J. Mészáros who connected various definitions of contractive mappings (cf. [27]). We also note that concave Young functions can meet the contractive property.

Theorem 5.2 (Agbeko, [11]) *Let $\Phi \in \mathcal{Y}_{\text{conc}}$ be arbitrary. In order that there be a constant $s > 0$ for which $\varphi(s) < 1$, it is necessary and sufficient that Φ admit a positive fixed point, i.e. $\Phi(x) = x$ for some number $x > 0$.*

Proposition 5.1 (Agbeko, [11]) *Let $\Phi \in \mathcal{Y}_{\text{conc}}$ be arbitrary. If $x_0 \in (0, \infty)$ is such that $\Phi(x_0) = x_0$, then $\varphi(x_0) < 1$.*

Definition 5.3 (Agbeko, [11]) *A number $s > 0$ is called the degree of contraction of a function $\Phi \in \mathcal{Y}_{\text{conc}}$ if $\varphi(s) = 1$.*

We note in this case that $\varphi(s + \delta) < 1$ for any positive number δ, which makes Φ a contraction for some suitable δ.

The degree of contraction can provide a starting point for any iteration for finding the positive fixed points of concave Young-functions. In this viewpoint the degree of contraction can be useful, as a matter of fact.

6 Applications: Algorithmic determination of optimal measures from data

6.1 The determination of optimal measure from data

6.1.1 Some preliminary

In fuzzy sets theory the crux was how to determine the values of the fuzzy measure in a given real problem. To achieve that goal the Sugeno integral was used alongside with the so-called genetic algorithm to solve it (see [40]), say. The Sugeno integral with respect to a given fuzzy measure μ is regarded as a multi-input single-output system. The input is the integrand, i.e. the vector $(f(\omega_1), \ldots, f(\omega_n))$, while the output is the value of its

Sugeno integral $E := (\mathrm{S})\int f d\mu = \sup\{\alpha \wedge \mu(F_\alpha) : \alpha \in [0,\, 1]\}$, where f is a measurable function defined on a finite measurable space $(\Omega,\, \mathcal{F})$ and $F_\alpha := \{\omega \in \Omega : f(\omega) \geq \alpha\}$. By repeatedly observing the system $(f(\omega_1),\, \dots,\, f(\omega_n))$ results the following

$$\begin{array}{|cccc|c|} \hline f_1(\omega_1) & f_1(\omega_2) & \dots & f_1(\omega_n) & E_1 \\ f_2(\omega_1) & f_2(\omega_2) & \dots & f_2(\omega_n) & E_2 \\ \vdots & \vdots & & \vdots & \vdots \\ f_k(\omega_1) & f_k(\omega_2) & \dots & f_k(\omega_n) & E_k \\ \hline \end{array}$$

and we look for an approximate fuzzy measure μ with $E_i = (\mathrm{S})\int f_i d\mu$, $(i = 1,\, \dots,\, k)$, such that the expression

$$e := \sqrt{\frac{1}{k}\sum\nolimits_{i=1}^{k}\left(E_i - (S)\int f_i d\mu\right)^2}$$

is minimized. For more about the genetic algorithm see [25], for example.

6.1.2 Problems

Problem 1 *Let $(\Omega,\, \mathcal{F})$ be the measurable space with $\Omega = \{1,\, \dots,\, n\}$ and $\mathcal{F} = 2^\Omega$. Write $B_1 := \{1\},\, \dots,\, B_n := \{n\}$ and let f be a random variable assuming the theoretical values in $[0,\, \infty)$. Observe k times this measurable function with results $f_1,\, \dots,\, f_k$, i.e.*

$$\begin{array}{|cccc|c|} \hline f_1(1) & f_1(2) & \dots & f_1(n) & Q_1 \\ f_2(1) & f_2(2) & \dots & f_2(n) & Q_2 \\ \vdots & \vdots & & \vdots & \vdots \\ f_k(1) & f_k(2) & \dots & f_k(n) & Q_k \\ \hline \end{array}$$

where $Q_i = \frac{1}{n}\sum_{j=1}^{n} f_{ij}$ with $f_{ij} := f_i(j)$, $j = 1,\, \dots,\, n$, and $i = 1,\, \dots,\, k$. The question is to know which one of these sample averages can "best" approximate the theoretical mathematical expectation.

To solve Problem 1 we propose to look for an approximation of the theoretical optimal measure p for which $\int_\Omega f_i dp \approx Q_i$, $(i = 1,\, \dots,\, k)$, such that the expression

$$err := \sqrt{\sum\nolimits_{i=1}^{k}\varepsilon_i{}^2} = \sqrt{\sum\nolimits_{i=1}^{k}\left(Q_i - \int_\Omega f_i dp\right)^2}$$

is minimized. Write p_{**} for the optimal measure p for which the least square is minimal. Now, it is not difficult to see that $\bigvee_{i=1}^{k} \left| Q_i - \int_\Omega f_i dp_{**} \right| < err$. Let i_0 be the index where the maximum is attained, i.e.

$$\left| Q_{i_0} - \int_\Omega f_{i_0} dp_{**} \right| = \bigvee_{i=1}^{k} \left| Q_i - \int_\Omega f_i dp_{**} \right|.$$

Then we can conclude that with respect to the optimal measure p_{**} the i_0th sample provides us with the best possible sample average.

As we know statistical spaces are not restricted in general to the real line nor to the real vector spaces. For this reason we shall formulate the following problem. We shall then indicate how to use the solution of first problem to solve the second one.

Problem 2 *Let $(X, \mathcal{S})$ be measurable space with $\mathcal{S}$ being an arbitrary σ-algebra. Fix a partion $D_1, \ldots, D_n$ of X and consider a random variable $h : X \to [0, \infty)$, assuming theoretical values. Observe k times this measurable function with results*

D_1	D_2	$\ldots$	D_n	
h_{11}	h_{12}	$\ldots$	h_{1n}	Q_1
h_{21}	h_{22}	$\ldots$	h_{2n}	Q_2
$\vdots$	$\vdots$		$\vdots$	$\vdots$
h_{k1}	h_{k2}	$\ldots$	h_{kn}	Q_k

where h_{ij} is the observed value of h in the ith experiment on event D_j, $i = 1, \ldots, k$; $j = 1, \ldots, n$, and $Q_i = \frac{1}{n}\sum_{j=1}^{n} h_{ij}$, $i = 1, \ldots, k$. The question is to know which one of these sample averages can "best" approximate the theoretical mathematical expectation of h.

To solve Problem 2, first write $\mathcal{S}_0 := \sigma(D_1, \ldots, D_n)$. We note that $\mathcal{S}_0$ is a finite σ-algebra and the random variable h is also $\mathcal{S}_0$-measurable. Clearly, $\mathcal{S}_0$ and 2^Ω are equinumerous, where $\Omega = \{1, \ldots, n\}$. Then Problem 2 can be reduced to Problem 1 if we define $f_{ij} := h_{ij}$, $i = 1, \ldots, k$; $j = 1, \ldots, n$.

6.2 Algorithm to solve the first problem

Step 0

Input: n positive integer
$\Omega = \{1, \dots, n\}$
$k \times n$ matrix $F = [f(i,j)]_{i,j=1}^{n,\,k}$
n-dimensional vector Q
error bound ε
$B_j = \{j\}$, $j = 1 \ldots n$
$X =$ the power set of Ω whose elements should be indexed $kk = 1 \ldots 2^n$

Step 1.

Generate a decreasing sequence $\alpha(j) \in (0,\, 1]$, with $\alpha(1) = 1$.

Step 2.

Permute $\sigma\left(\{1,\, \ldots,\, n\}\right) = \{n_1,\, \ldots,\, n_n\}$
Put $p\left(B_j\right) = \alpha(n_j)$, for $j = 1,\, \ldots,\, n$
Compute the optimal average: $A(i) = \max\{f(i,j) * p\left(B_j\right) : j = 1 \ldots n\}$
Compute the corresponding error: $err = \sqrt{\left(\sum_{j=1}^{n} \left(Q(i) - A(i)\right)\right)^2}$

$iter = 1$
Step 3.

If $err < \varepsilon$ or $iter > n!$ *do*
Find the index i_0: $|Q(i_0) - A(i_0)| = \max\{|Q(i) - A(i)| : i = 1 \ldots k\}$
Determine $p\left(B\right) = \max\left\{\alpha(n_j) : j \in B\right\}$, for each $B \in X$
Else GOTO **Step 2**
Step 4.

The outputs
1.) Best sample: $f\left(i_0,\, 1\right),\, \ldots,\, f\left(i_0,\, n\right)$
2.) The approximated optimal measure:

2^{Ω}	$p\left(B\right)$
$\{\}$	0
B_1	$p\left(B_1\right)$
$\vdots$	$\vdots$
B_i	$p\left(B_i\right)$
$\vdots$	$\vdots$

6.3 An algorithm to find the degree of contraction and the fixed point

Step 1. Input $\Phi(x)$, $cc > 0$.

Step 2. Compute the derivative $\varphi(x)$ of $\Phi(x)$

Step 3. Starting from cc find an approximation root for equation $\varphi(x) - 1 = 0$ and put the result into c.

Step 4. *If* $c = 0$ *then* STOP.
else do

Step 5. Starting from c apply the FixedPoint algorithm, i.e.
$x_0 := c;\ x_{k+1} := \Phi(x_k);\ k = k + 1.$

List of independent references

[1] I. FAZEKAS, A note on "optimal measures", Publ. Math. Debrecen **51** / 3-4 (1997), 273-277. (**Paper referred to:** Publ. Math. Debrecen **46** / **1-2** (1995), 79-87)

[2] I. FAZEKAS, A note on "optimal measures", Publ. Math. Debrecen **51** / 3-4 (1997), 273-277. (**Paper referred to:** Acta Math. Hung. **63** (**1-2**) (1994), 1–15.)

[3] TASOS C. CHRISTOFIDES, Maximal inequalities for N-demimartingales, Archives Ineq. Applic. 1(2003), 397 - 408. (**Paper referred to:** Ann. Univ. Sci. Budapest, Sect. Math. **29** (1986), 9-17)

[4] B.L.S. PRAKASA RAO, On some maximal inequalities for demisubmartingales and N-demisuper martingales, J. Inequal. Pure and Appl. Math. **8**(2006), Issue 4, Art.112, pp. 17 [ONLINE: http://jipam.vu.edu.au/] (**Paper referred to:** Ann. Univ. Sci. Budapest, Sect. Math. **29** (1986), 9-17)

[5] YU MIAO AND GUANGYU YANG, A note on the upper bounds for the dispersion, J. Inequal. Pure and Appl. Math. **8**(2007), Issue 3, Art.83, 17 pp. 6. [ONLINE: http://jipam.vu.edu.au/] (**Paper referred to:** Inequal. Pure and Appl. Math. **7**(5) (2006), Art.186. [ONLINE: http://jipam.vu.edu.au/])

References

[1] N. K. Agbeko, *Some reverse maximal inequalities for supermartingales*, Ann. Univ. Sci. Budapest, Sect. Comput. **6** (1985), 49–54.

[2] N. K. Agbeko, *Concave function inequalities for sub- (super-) martingales*, Ann. Univ. Sci. Budapest, Sect. Math. **29** (1986), 9–17.

[3] N. K. Agbeko, *Necessary and sufficient condition for the maximal inequality of concave Young-functions*, Ann. Univ. Sci. Budapest, Sect. Math. **32** (1989), 267–270.

[4] N. K. Agbeko, *On an inequality of Longnecker and Serfling*, Analysis Math. **17** (1991), 3–9.

[5] N. K. Agbeko, *On optimal averages*, Acta Math. Hung. **63** (**1-2**) (1994), 1–15.

[6] N. K. Agbeko, *On the structure of optimal measures and some of its applications*, Publ. Math. Debrecen **46**/**1-2** (1995), 79–87.

[7] N. K. Agbeko, *How to characterize some properties of measurable functions,* Math. Notes, Miskolc **1**/**2** (2000), 87–98.

[8] N. K. Agbeko, *Mapping bijctively σ-algebras onto power sets,* Math. Notes, Miskolc **2**/**2** (2001), 85–92.

[9] N. K. Agbeko, *Studies on concave Young-functions* , Miskolc Math. Notes **6**/**1** (2005), 3–18.

[10] N. K. Agbeko, *Some p.d.f.-free upper bound for the dispersion $\sigma(X)$* and the quantity $\sigma^2(X) + (x - EX)^2$, J. Inequal. Pure and Appl. Math. **7**(5) (2006), Art.186. [ONLINE: http://jipam.vu.edu.au/]

[11] N. K. Agbeko, *The class of concave Young-functions possessing a positive fixed point*, Miskolc Math. Notes **9** (2008), No. 1, pp. 3-6.

[12] R. Ash, *Real analysis and probability theory*, Acad. Press, New York, London (1972).

[13] N. S. Barnett, P. Cerone, S. S. Dragomir and J. Roumeliotis, *Some inequalities for the dispersion of a random variable whose pdf is defined on a finite interval*, J. Inequal. Pure and Appl. Math., **2**(1)(2001), Art. 1. [ONLINE: http://jipam.vu.edu.au]

[14] L. E. J. Brouwer, *Über Abbildungen von Mannigfaltigkeiten*, Math. Ann., **71**(1912), 97–115.

[15] Tasos C. Christofides, *Maximal inequalities for N-demimartingales*, Archives Ineq. Applic. 1(2003), 397 - 408.

[16] Á. Császár and M. Laczkovich, *Discrete and equal convergence*, Studia Sci. Math. Hung. **10**(1975), 463–472.

[17] Á. Császár and M. Laczkovich, *Some remarks on discrete Baire classes,* Acta Math. Acad. Sci. Hung. **33 (1-2)**(1979), 51–70.

[18] Á. Császár and M. Laczkovich, *Discrete and equal Baire classes*, Acta Math. Hung. **55(1-2)** (1990), 165–178.

[19] D. Dubois, *Théorie des possibilités*, Masson, Paris, 1985. **ISBN: 2-225–80579-2**

[20] I. Fazekas**,** *A note on "optimal measures"*, Publ. Math. Debrecen **51/3-4** (1997), 273–277.

[21] P. R. Halmos, *Measure theory*, D. van Nostrand Co. Inc. 4^{th} ed., 1956.

[22] Shizuo Kakutani, *A generalization of Brouwer's fixed point theorem*, Duke Mathematical Journal **8**(3)(1941), 457-459.

[23] J. Kennan, *Uniqueness of positive fixed points for increasing concave functions on $\mathbb{R}^n$: An elementary result*, Review of Econmic Dynamics **4**(2001), 893-899.

[24] M. A. Krasnosel'ski and B. Ya Rutickii, *Convex functions and Orlicz-spaces.* (Transl. from Russian by Boron L. F.) Noordhoff, Groningen, 1961.

[25] C. C. Lin and A. P. Chen, *Fuzzy discrimnimant analysis with outlier detection by genetic algorithm*, Computer & Operation Reaserch **31** (2004), 877-888.

[26] M. Longnecker and R. J. Serflying, *General moment and probability inequalities for the maximum partial sum*, Acta Math. Acad. Sci. Hung. **30** (1977), 129–133.

[27] J. Mészáros, *A comparison of various definitions of contractive type mappings*, Bull. Cal. Math. Soc. **84**(1992), 167-194.

[28] Yu Miao and Guangyu Yang, *A note on the upper bounds for the dispersion*, J. Inequal. Pure and. Appl. Math., 8(3)(2007), Art. 83 [ONLINE: http://jipam.vu.edu.au/].

[29] J. Mogyoródi, *On the decomposition of Doob of non-negative submartingales*, Ann. Univ. Sci. Budapest, Sectio Math. **24** (1981), 255–264.

[30] J. Mogyoródi and T. F. Móri, *Necessary and sufficient condition for the maximal inequality of convex Young functions*, Acta Sci. Math. Szeged **45** (1983), 325–332.

[31] J. Mogyoródi, *On a concave Young function inequality for martingales*, Annales Univ. Sci. Budapest, Sectio Math. **24** (1981), 265–271.

[32] J. F. Nash, Jr., *Equilibrium Points in N-Person Games*, Proc. Nat. Acad. Sci. U.S.A. **36**(1950) 48–49.

[33] B.L.S. Prakasa Rao, *On some maximal inequalities fo demisubmartingales and N-demisupermartingales*, J. Inequal. Pure and. Appl. Math., 8(4)(2007), Art. 112. [ONLINE: http://jipam.vu.edu.au/].

[34] M. Puri and D. Ralescu, *A possibility measure is not a fuzzy measure*, Fuzzy Sets and Systems, **7**(1982), 311-313.

[35] M. C. Reed, *Fundamental Ideas of Analysis*, John Wiley and Sons Inc., New York, Singapore, 1998.

[36] W. Rudin, *Principles of mathematical analysis*, 3rd ed., McGraw-Hill Book Company, Auckland, 1976.

[37] J. Serfling, *Moment inequality for the maximum cumulative sum*, Ann. Math. Statist. **41** (1970), 1227–1234.

[38] M. Sugeno and T. Murofushi, *Pseudo-additive measures and integrals*, J. Math. Anal. Appl. **122**(1987), **no. 1**, 197-222.

[39] A. Tarski, *A lattice-theoretical Fixpoint Theorem and its applications*, Pacific Journal of Mathematics **5**(1955), 285-309.

[40] Z. Wang, K.-S. Leung and J. Wang, *Determining nonnegative monotone set functions based on Sugeno's integral: an application of genetic algorithms*, Fuzzy Sets and Systems, **112**(2000), 155-164.

Summary

In the image of the probability measure we proposed a set function, called optimal measure, which we showed to have a structural property. Next, we defined the so-called optimal average for non-negative measurable simple functions and then extend this definition to arbitrary measurable functions. Optimal average provides us with many similar well-known results in measure theory such as the Fubini and Radon-Nikodym theorems, say. We characterized various notions of well-known convergence such as the notions of discrete, equally, uniform and pointwise convergence of sequences of measurable functions. Maximal inequalities in connection with concave (convex) Young functions are discussed are studied with probabilistic tools. Further we isolated a subset $\mathfrak{A}$ of the set $\mathcal{Y}_{\mathrm{conc}}$ of concave Young functions and showed that it is closed under the composition operation. We also demonstrated that subset $\mathfrak{A}$ is a dense set in $\mathcal{Y}_{\mathrm{conc}}$ with respect to a specific metric. Finally we characterized the set of those concave Young functions possessing a positive fixed point.

Összefoglaló

Bevezettük az optimális mértéket és beláttuk, hogy minden optimális mérték strukturális tulajdonságú. Definiáltuk az optimális átlagot nem-negatív mérhető lépcsős függvények esetén. Először beláttuk, hogy ez az átlag nem függ a lépcsős függvény felbontásától. Kiterjesztettük az optimális átlagot tetszőleges nem-negatív mérhető függvényekre és így megkaptuk a Fubini, illetve Radon-Nikodym tételek a mértékelméletbeli megfelelőjét. Jellemeztünk számos jól ismert konvergencia fogalmakat a mérhető függvénysorozatok esetén: a stabilizálódó, egy sorozat szerint pontonkénti, egyenletes, valamint a pontonkénti konvergencia fogalmakat. Kitértünk a mérhető függvénysorozatok különféle korlátosságának jellemzésére is az optimális átlag alkalmazásával. Áttekintjük a konkáv (konvex) Young függvényekkel kapcsolatos maximális egyenlőtlensége-ket valószínűségszámítási eszközökkel. Elkülönítettük az $\mathcal{Y}_{\mathrm{conc}}$ konkáv Young függvények halmazának egy $\mathfrak{A}$ részhalmazát, mely a kompozícióra zárt. Megadtunk az $\mathcal{Y}_{\mathrm{conc}}$ halmazon egy olyan metrikát, mely szerint az $\mathfrak{A}$ részhalmaz sűrű az $\mathcal{Y}_{\mathrm{conc}}$ halmazban. Megadtuk a pozitív fixponttal rendelkező összes konkáv Young függvények halmazát.