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The role of the maximum operator in the theory of measurability and some applications

PhD Thesis

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PREFACE

The present thesis aims to conduct some studies of the maximum (supremum) operator on:

i.) σ-algebras, i.e. to define on σ-algebras functions which map the union into the maximum under certain restrictions opening the door to the characterization of mapping bijectively σ-algebras onto power sets,

ii.) the set of measurable functions with various characterizations

The material is presented essentially in seven chapters almost all of which begin with an introductory part. In the first some historical backgrounds are presented. Chapters II–V deal essentially with results in connection with optimal measure which is a function, continuous from above and suitably normalized, mapping any given σ-algebra into the interval such that every finite union is mapped into a maximum. We point out that the choice of the term *optimal measure* is deliberate, since taking the maximum also encounters the meaning given in the *Oxford Dictionary* to the world "optimal". Chapter VI treats some maximal inequalities regarding random variables. In the last chapter we present some applications.

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CHAPTER I

HISTORICAL BACKGROUNDS

Notations.

* $\mathbb{N}$ denotes the set of positive integers.
* $\mathbb{R}$ denotes the set of real numbers.
* $\mathbb{R}_+$ denotes the set of non-negative real numbers.
* $\chi(B)$ stands for the characteristic function of the set B.
* $|B|$ designates the cardinality of the set B.
* $\bigvee$ and $\vee$ (respectively, $\bigwedge$ and $\wedge$) stand for the maximum (respectively the minimum) operator.
* $\mathcal{P} := \mathcal{P}_{<\infty} \cup \mathcal{P}_{\infty}$ will denote the set of all optimal measures defined on measurable space $(\Omega, \mathcal{F})$, with both Ω and $\mathcal{F}$ being infinite sets, where $\mathcal{P}_{<\infty}$ (resp. $\mathcal{P}_{\infty}$) denotes the set of all optimal measures whose generating systems are finite (resp. countably infinite).
* For every $A \in \mathcal{F}$, we write $\overline{A}$ for the complement of A.
* $A \subset B$ means set A is a proper subset of set B.
* $A \subseteq B$ means set A is a subset of set B.
* The power set of set A will be denoted by $\mathbb{P}(A)$ or 2^A.

1.1 About the convergence of function sequences

Augustin Louis Cauchy in 1821 published a faulty proof of the false statement that the pointwise limit of a sequence of continuous functions is always continuous. Joseph Fourier and Niels Henrik Abel found counter examples in the context of Fourier series. Dirichlet then analyzed Cauchy's proof and found the mistake: the notion of pointwise convergence had to be replaced by uniform convergence.
The concept of uniform convergence was probably first used by Christoph Gudermann. Later his pupil Karl Weierstrass coined the term *gleichmäßig konvergent* (German: uniform convergence) which he used in his 1841 paper *Zur Theorie der Potenzreihen*, published in 1894. Independently a similar concept was used by Philipp Ludwig von Seidel and George Gabriel Stokes but without having any major impact on further development. G. H. Hardy compares the three definitions in his paper *Sir George Stokes and the concept of uniform convergence* and remarks: *Weierstrass's discovery was the earliest, and he alone fully realized its far-reaching importance as one of the fundamental ideas of analysis.* For more materials about these facts we refer to [63] or
http://en.wikipedia.org/wiki/Uniform_convergence.

Ever since many other types of convergence have been brought to light. We can list some few of them: discrete and equal convergence introduced by Á. Császár and M. Laczkovich in 1975 (cf. [27, 28, 29]), topologically speaking the weak and strong convergence, the latest being at the origin of the so-called *Banach spaces*, which are very broad and interesting classes of functions, indeed.

1.2 Outer measure

The question "*Can we assign to a subset B of $\mathbb{R}$ a measure of its length?*" had been of great importance. The answer to this problem lies in measure theory, a subject that was pioneered by Lebesgue, Borel and others at the beginning of the *20th* century and which proved to have an immense impact on modern analysis and probability theory, as well as on many other areas of mathematics.

Definition 1.2.1 *An outer measure is an extended real-valued set function μ^* having the following properties:*

- **i.)** *The domain of definition of μ^* consists of all the subsets of a set $X \neq \varnothing$.*
- **ii.)** *μ^* is non-negative.*
- **iii.)** *μ^* is countably subadditive, i.e.*

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*\left(A_n\right)$$

  whenever (A_n) is a sequence of subsets of X.
- **iv.)** *μ^* is monotone.*
- **v.)** *$\mu^*(\varnothing) = 0$.*

Definition 1.2.2 *Given an outer measure μ^*, we say that a set E is μ^*-measurable if $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \backslash E)$ for any subset $A \subset X$.*

The (vague) motivation for Definition 1.2.2 is that the sets we want to single out as μ^*-measurable should be such that μ^* will be additive on them.

Theorem 1.2.1 *Let μ^* be an outer measure and denote by $\mathcal{A}$ the class of all μ^*-measurable sets of a set $X \neq \varnothing$. Then*

- **i.)** $X \in \mathcal{A}$.
- **ii.)** $\overline{B} \in \mathcal{A}$ whenever $B \in \mathcal{A}$ (where $\overline{B}$ denotes the complement of set B).
- **iii.)** $(B_n) \subset \mathcal{A}$, then $\bigcup_{n=1}^{\infty} B_n \in \mathcal{A}$.

Moreover, if μ denotes the restriction of μ^* to $\mathcal{A}$, then

- **iv.)** $\mu(\varnothing) = 0$, $\mu(B) \geq 0$ whenever $B \in \mathcal{A}$,
- **v.)** $\mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n)$ for every sequence $(B_n) \subset \mathcal{A}$ whose members are pairwise disjoint, in this case μ is commonly reported to be σ-*additive.*

Every collection $\mathcal{A}$ of sets meeting properties (i)–(iii) is called σ-*algebra* and every set function μ satisfying properties (iv) and (v) is referred to as *measure.*

1.3 Construction of outer measures

Let $\mathcal{K}$ be a class of subsets of a set $X \neq \varnothing$. We call $\mathcal{K}$ a sequential covering class (of X) if:

i.) $\varnothing \in \mathcal{K}$.

ii.) For every set A there is a sequence $(B_n) \subset \mathcal{K}$ such that $A \subset \bigcup_{n=1}^{\infty} B_n$.

For example the bounded open intervals on the real line form a sequential covering class of $\mathbb{R}$.

Let λ be an extended real-valued, non-negative set function, with domain $\mathcal{K}$, such that $\lambda(\varnothing) = 0$. For each subset A of X let

$$\mu^*(A) = \inf\left\{\sum_{n=1}^{\infty} \lambda(B_n) : (B_n) \subset \mathcal{K},\ A \subset \bigcup_{n=1}^{\infty} B_n\right\} \tag{1}$$

Theorem 1.3.1 *For any sequential covering class $\mathcal{K}$ and for any non-negative, extended real-valued set function λ with domain $\mathcal{K}$ and $\lambda(\varnothing) = 0$, the set function μ^* defined by* (1) *is an outer measure.*

1.4 Completion of measures

A measure μ with domain $\mathcal{A}$ is said to be *complete* if for any two sets N, E the following holds: *If $N \subset E$, $E \in \mathcal{A}$ and $\mu(E) = 0$, then $N \in \mathcal{A}$.*

Note that the measure constructed in Theorem 1.2.1 is complete.

Theorem 1.4.1 *Let μ be a measure on a σ-algebra $\mathcal{A}$ and let $\overline{\mathcal{A}}$ denote the class of all sets of the form $E \cup N$, where $E \in \mathcal{A}$ and N is any subset of a set of $\mathcal{A}$ of measure zero. Then $\overline{\mathcal{A}}$ is a σ-algebra and the set function $\overline{\mu}$ defined by $\overline{\mu}(E \cup N) = \mu(E)$ is a complete measure on $\mathcal{A}$.*

1.5 Lebesgue measures

Denote by $\mathbb{R}^n$ the Euclidean space of n dimensions. The points of $\mathbb{R}^n$ are written in the form $x = (x_1, \ldots, x_n)$. By an *open interval* we shall mean a set of the form

$$I_{a,b} := \{x = (x_1, \ldots, x_n) : a_i < x_i < b_i \text{ for } i = 1, \ldots, n\}$$

where $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$.

The set $\mathcal{K}$ of all open intervals forms a sequential covering class of $\mathbb{R}^n$. Let λ be given by $\lambda(\varnothing) = 0$ and $\lambda(I_{a,b}) = \prod_{j=1}^{n}(b_j - a_j)$, if $a \neq b$. The outer measure determined by the pair $\mathcal{K}$, λ (in accordance with Theorem 1.3.1) is called the *Lebesgue outer measure*. The complete measure determined by this outer measure (in accordance with Theorem 1.2.1) is called the *Lebesgue measure*. The measurable sets are called the *Lebesgue-measurable sets*.

If the *axiom of choice* is agreed upon, then the following important result is a valid argument.

Theorem 1.5.1 (P. R. Halmos, [37]) *There exists a set on the real line that is not Lebesgue-measurable.*

1.6 Notion of abstract measure and probability theories

Definition 1.6.1 *A collection $\mathcal{F}$ of subsets of a set $\Omega \neq \varnothing$ is called σ-algebra if:*

i.) $\overline{A} \in \mathcal{F}$ *whenever* $A \in \mathcal{F}$.

ii.) $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$ *whenever* $(A_n) \subset \mathcal{F}$.

The pair $(\Omega, \mathcal{F})$ is then referred to as a *measurable space* and every member of $\mathcal{F}$ is called a *measurable set*.

The triple $(\Omega, \mathcal{F}, \mu)$ is called *a measure space* if $(\Omega, \mathcal{F})$ is a measurable space and the function $\mu : \mathcal{F} \to [0, \infty)$ is a *measure*, in the sense that μ meets the following two properties:

1.) $\mu(\varnothing) = 0$.

2.) $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$ whenever $(A_n) \subset \mathcal{F}$ is a sequence of pairwise disjoint measurable sets.

By a *measurable function* defined on the measurable space $(\Omega, \mathcal{F})$, we mean a function $f : \Omega \to \mathbb{R}$ for which $(f < b) \in \mathcal{F}$ for all $b \in \mathbb{R}$.

Two measurable functions f and g are said to be *equal almost everywhere* if $\mu(f \neq g) = 0$. This is manifestly an equivalence relation, and we remind that if f is a measurable function, then it is let to coincide with the induced equivalence class.

We note that various types of convergence (for instance the pointwise) of sequences of measurable functions are widely treated in the literature.

The basic notion of modern probability theory, pioneered by the Russian Mathematician A. N. Kolmogorov, is an adaptation, in a sense, of measure theory. This idea proved to be very genius, since many other probabilistic theories, such as the theory of stochastic processes, martingale theory, mathematical statistics and so on have been brought to light. Though probability theory is derived from analysis there are specific notions that cannot be treated by means of analysis: the notion of independence, conditional probability and conditional expectation, which is guaranteed by the *Radon-Nikodym theorem*. One of the areas of interest of the present candidate is martingale theory. What exactly martingale is? Before reminding this definition let us first refresh our mind over the conditional expectation (cf. [53]).

Definition 1.6.2 *Let X be a random variable with finite expectation on a probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{S} \subset \mathcal{F}$ a sub-σ-algebra. The conditional expectation of X given $\mathcal{S}$, is a random variable to be denoted by $E(X|\mathcal{S})$, such that $E(X|\mathcal{S})$ is an $\mathcal{S}$-measurable function and $E(\chi_A E(X|\mathcal{S})) = E(X\chi_A)$ for every $A \in \mathcal{S}$.*

Definition 1.6.3 *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(\mathcal{F}_n) \subset \mathcal{F}$ an increasing sequence of σ-algebras. The pair $(X_n, \mathcal{F}_n)$, $n \in \mathbb{N}$, is referred to as a submartingale if for every $n \in \mathbb{N}$ the following conditions hold simultaneously*:

a.) $EX_n < \infty$,

b.) X_n *is* $\mathcal{F}_n$*-measurable,*

c.) $E(X_{n+1}|\mathcal{F}_n) \geq X_n$ *with probability one. Whenever the inequality in* **(c)** *is reversed we then speak of supermartingale.*

If $(X_n, \mathcal{F}_n)$, $n \in \mathbb{N}$, is both a submartingale and a supermartingale, then we speak of martingale (cf. [52]).

J. L. Doob is actually the initiator of martingale theory. Perhaps, we should remind one of the most fundamental results he obtained, called Doob's maximal inequality (cf. [52]):

Theorem 1.6.1 (Doob's inequality) *Let $(X_n, \mathcal{F}_n)$ $n \in \mathbb{N}$, be a non-negative submartingale. Then for every number $x > 0$, we have*

$$xP(X_n^* \geq x) \leq EX_n\chi(X_n^* \geq x),$$

where $X_n^* := \bigvee_{k=1}^{n} X_k$.

1.7 Maxitive or possibility measures and integration operators

In another imitation of measure and Lebesgue integral, the so-called maxitive measure and a corresponding integral were introduced by N. Shilkret (cf. [62]). This led to the birth of the theory of fuzzy set.

Definition 1.7.1 *Let $\mathcal{R}$ be a σ-ring of subsets of an arbitrary set Ω. An extended non-negative real valued function m on $\mathcal{R}$ is called a maxitive measure if $m(\varnothing) = 0$ and*

$$m\left(\bigcup_{i \in I} E_i\right) = \sup_{i \in I} m(E_i)$$

for any collection of pairwise disjoint sets $\{E_i : i \in I\} \subset \mathcal{R}$, where I denotes an arbitrary countable index set.

The functional $\int f := \sup_{b \geq 0} bm(f \geq b)$ was defined to replace the Lebesgue integral, accordingly.

In [67, 64, 55] the notions of fuzzy sets as well as pseudo-additive and fuzzy measures were initiated as follows.

Definition 1.7.2 *Let* $(\Omega, \mathcal{F})$ *be measurable space. A set function* $\mu : \mathcal{F} \to [0, 1]$ *is said to be a fuzzy measure if and only if the conditions here below hold*:

(F1) *The identity* $\mu(\varnothing) = 0$ *holds.*

(F2) *Whenever* $A, B \in \mathcal{F}$ *and* $A \subset B$, *then* $\mu(A) \leq \mu(B)$.

(F3) *Whenever* $(A_n) \subset \mathcal{F}$ *and* $A_n \uparrow A$, *then* $\mu(A_n) \uparrow \mu(A)$.

(F4) *Whenever* $(A_n) \subset \mathcal{F}$ *and* $B_n \downarrow B$, *then* $\mu(B_n) \downarrow \mu(B)$.

A *fuzzy integral* of a measurable function $h : \Omega \to [0, 1]$ is defined by

$$(\mathrm{S})\int h d\mu := \sup_{x\in[0,1]} \min\{x, \mu(\{\omega : h(\omega) > x\})\}$$

and is often called the *Sugeno integral.*

We need to notice that fuzzy measure is a generalization of both probability and optimal measures, because they meet the above four axioms.

At times fuzzy measure is defined by the collection of the following axioms:

Definition 1.7.3 *Let* $(\Omega, \mathcal{F})$ *be measurable space. A set function* $\mu : \mathcal{F} \to [0, 1]$ *is said to be a fuzzy measure if and only if the conditions here below are met simultaneously:*

(S1) *The identity* $\mu(\varnothing) = 0$ *holds.*

(S2) *If* $A, B \in \mathcal{F}$ *and* $A \subset B \Longrightarrow \mu(A) \leq \mu(B)$.

(S3) *If* $A, B \in \mathcal{F}$ *and* $A \cap B = \varnothing$, *then* $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$.

(S4) *If* $(A_n) \subset \mathcal{F}$ *and* $A_n \uparrow A$, *then* $\mu(A_n) \uparrow \mu(A)$.

In fact, we should note that this form of fuzzy measure inspired the candidate in laying down the theory of optimal measure.

CHAPTER II

OPTIMAL MEASURES AND THE STRUCTURE THEOREM

2.1 Introduction

This section can be seen at the beginning of the work [5].

Definition 2.1.1 *A set function* $p : \mathcal{F} \to [0,\, 1]$ *will be called optimal measure if it satisfies the following three axioms:*

Axiom 1. $p\,(\Omega) = 1$ *and* $p\,(\varnothing) = 0$.

Axiom 2. $p\,(B \cup E) = p\,(B) \vee p\,(E)$ *for all measurable sets* B *and* E.

Axiom 3. p *is continuous from above, i.e. whenever* $(E_n) \subset \mathcal{F}$ *is a decreasing sequence, then* $p\left(\bigcap\limits_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} p\,(E_n) = \bigwedge\limits_{n=1}^{\infty} p\,(E_n)$.

The triple $(\Omega,\, \mathcal{F},\, p)$ will be referred to as an *optimal measure space.* For all measurable sets B and C with $B \subset C$, the identity

$$p\,(C \backslash B) = p\,(C) - p\,(B) + \min\{p\,(C \backslash B)\,,\, p\,(B)\} \tag{2}$$

holds, and especially for all $B \in \mathcal{F}$,

$$p\left(\overline{B}\right) = 1 - p\,(B) + \min\left\{p\,(B)\,,\, p\left(\overline{B}\right)\right\}.$$

In fact, it is obvious (via *Axiom 2.1*) that,

$$\begin{aligned} p\,(B) + p\,(C \backslash B) &= \max\{p\,(C \backslash B)\,,\, p\,(B)\} + \min\{p\,(C \backslash B)\,,\, p\,(B)\} \\ &= p\,(C) + \min\{p\,(C \backslash B)\,,\, p\,(B)\}\,. \end{aligned}$$

Lemma 2.1.1 *Let* $(B_n) \subset \mathcal{F}$ *be any sequence tending increasingly to a measurable set* B*, and* p *an optimal measure. Then* $\lim_{n\to\infty} p\,(B_n) = p\,(B)$*.*

Proof. The lemma will be proved if we show that for some $n_0 \in \mathbb{N}$, the identity $p\,(B) = p\,(B_n)$ holds true whenever $n \geq n_0$. Assume that for every $n \in \mathbb{N}$, $p\,(B) \neq p\,(B_n)$, which is equivalent to $p\,(B_n) < p\,(B)$, for all $n \in \mathbb{N}$. This inequality, however, implies that $p\,(B) = p\,(B \backslash B_n)$ for each $n \in \mathbb{N}$. But since sequence $(B \backslash B_n)$ tends decreasingly to $\varnothing$, we must have that $p\,(B) = 0$, a contradiction which proves the lemma. ■

It is clear that every optimal measure p is monotonic and σ-subadditive. We would like to mention that the following example is essentially due to M. Laczkovich.

Example 2.1.1 *Let* $(\Omega, \mathcal{F})$ *be a measurable space,* $(\omega_n) \subset \Omega$ *be a fixed sequence, and* $(\alpha_n) \subset [0, 1]$ *a given sequence tending decreasingly to zero. The function* $p : \mathcal{F} \longrightarrow [0, 1]$, *defined by*

$$p(B) = \max\{\alpha_n : \omega_n \in B\} \tag{3}$$

is an optimal measure.
Moreover, if $\Omega = [0, 1]$ *and* $\mathcal{F}$ *is a* σ*-algebra of* $[0, 1]$ *containing the Borel sets, then every optimal measure defined on* $\mathcal{F}$ *can be obtained as in* (3).

Proof of the moreover part. We first prove that if $B \in \mathcal{F}$ and $p(B) = c > 0$, then there is an $x \in B$ which satisfies $p(\{x\}) = c$. To do this let us show that there exists a nested sequence of intervals $I_0 \supset I_1 \supset I_2 \supset \ldots$ such that $|I_n| = 2^{-n}$ and $p(B \cap I_n) = c$, for every $n \in \mathbb{N} \cup \{0\}$. In fact, let $I_0 = [0, 1]$. If I_n has been defined then let $I_n = E \cup H$, where E and H are non-overlapping intervals with $|E| = |H| = 2^{-n-1}$. Obviously, we may choose $I_{n+1} = E$ or H. By the continuity from above we have $p\left(\bigcap_{n=1}^{\infty}(B \cap I_n)\right) = c > 0$. In particular, $B \cap \left(\bigcap_{n=1}^{\infty} I_n\right) \neq \varnothing$. This implies that $B \cap \left(\bigcap_{n=1}^{\infty} I_n\right) = \{x\}$ and $p(\{x\}) = c$. Fix $c > 0$. Then the set $\{x : p(\{x\}) \geq c\}$ is finite. Assume in the contrary that there is an infinite sequence $(x_k) \subset [0, 1]$ such that $p(\{x_k\}) \geq c$, $k \in \mathbb{N}$. Thus denoting $B_k = \{x_k, x_{k+1}, \ldots\}$, it is clear that $\bigcap_{k=1}^{\infty} B_k = \varnothing$; but this contradicts the fact that $p(B_k) \geq c$. Consequently, the set $E_n = \{x : p(\{x\}) \geq n^{-1}\}$ is finite for all $n \in \mathbb{N}$. Hence there is a sequence $(x_n) \subset [0, 1]$ such that $p(\{x_n\}) \downarrow 0$ (as $n \to \infty$) and every point $x \in [0, 1]$ with $p(\{x\}) \geq 0$ is contained in (x_n). Therefore, for all $B \in \mathcal{F}$, $p(B) = \max\{\alpha_n : x_n \in B\}$ which is just the above optimal measure. ■

Example 2.1.2 *Let* $(\Omega, \mathcal{F})$ *be a measurable space. Clearly, if a function* $p_0 : \mathcal{F} \longrightarrow \{0, 1\}$ *is a* σ*-additive measure, then* $p_0(B \cup C) = p_0(B) + p_0(C) = \max\{p_0(B), p_0(C)\}$ *for all* B *and* $C \in \mathcal{F}$. *Hence* p_0 *is an optimal measure. One can easily show that* p_0 *is the only set function which is at the same time a* σ*-additive and optimal measure.*

Remark 2.1.1 *The collection* $M = \{B \in \mathcal{F} : p(B) < p(\Omega)\}$ *is a* σ*-ideal, whenever* p *is an optimal measure.*

2.2 The structure of optimal measures

To begin with, we note that the present section is entirely composed on the basis of paper [6].

By a p-atom we mean a measurable set $H, p(H) > 0$ such that whenever $B \in \mathcal{F}$ and $B \subset H$, then $p(B) = p(H)$ or $p(B) = 0$.

Definition 2.2.1 (Agbeko, [6]) *A* p*-atom* H *is decomposable if there exists a subatom* $B \subset H$ *such that* $p(B) = p(H) = p(H \backslash B)$. *If no such subatom exists, we shall say that* H *is indecomposable.*

Lemma 2.2.1 (Agbeko, [6]) *Any atom* H *can be expressed as the union of finitely many disjoint indecomposable subatoms of the same optimal measure as* H.

Proof. We say that a measurable set E is *good* if it an be expressed as the union of finitely many disjoint indecomposable subatoms. Let H be an atom and suppose that H is not good. Then H is decomposable. Set $H = B_1 \cup C_1$, where B_1 and C_1 are disjoint measurable sets with $p(B_1) = p(C_1) = p(H)$. Since H is not good, at least one of the two measurable sets B_1 and C_1 is not good; suppose, e.g. that B_1 is not good. Then B_1 is decomposable. Write $B_1 = B_2 \cup C_2$, where B_2 and C_2 are disjoint measurable sets with $p(B_2) = p(C_2) = p(H)$. Continuing this process for every $n \in \mathbb{N}$ we obtain two measurable sets B_n and C_n such that the C_n's are pairwise disjoint with $p(C_n) = p(H)$. This, however, is impossible since $E_n = \bigcup_{k=n}^{\infty} C_k$ tends decreasingly to the empty set and hence, by *Axiom 2.1*, $p(E_n) \to p(\varnothing)$ as $n \to \infty$, which contradicts that $p(E_n) \geq p(C_n) = p(H) > 0$, $n \in \mathbb{N}$. ■

An immediate consequent of *Lemma 2.2.1* is as follows.

Remark 2.2.1 *Let H be any indecomposable p-atom and E any measurable set, with $p(E) > 0$. Then, either $p(H) = p(H \backslash E)$ and $p(H \cap E) = 0$, or $p(H) = p(H \cap E)$ and $p(H \backslash E) = 0$.*

The Structure Theorem (Agbeko, [6]) *Let $(\Omega, \mathcal{F}, p)$ be an optimal measure space. Then there exists a collection $\mathcal{H}(p) = \{H_n : n \in J\}$ of disjoint indecomposable p-atoms, where J is some countable (i.e. finite or countably infinite) index set, such that for every measurable set $B \in \mathcal{F}$ with $p(B) > 0$ we have*

$$p(B) = \max\{p(B \cap H_n) : n \in J\}. \tag{4}$$

Moreover, if J is countably infinite, then the only limit point of the set $\{p(H_n) : n \in J\}$ is 0.

(Before we tackle the proof, let us state the following results.)

Lemma 2.2.2 *Let $E \in \mathcal{F}$ be with $p(E) > 0$, and $B_k \in \mathcal{F}$, $B_k \subset E$ $(k \in J)$, where J is any countable index set. Then*

$$p\left(\bigcup_{k \in J} B_k\right) < p(E) \tag{5}$$

if and only if

$$p(B_k) < p(E) \tag{6}$$

for all $k \in J$.

Proof. The lemma is obvious if the index set J is finite. Without loss of generality we may assume that $J = \mathbb{N}$. Suppose that (6) holds for all $k \in \mathbb{N}$. Put $C_k = \bigcup_{j=1}^{k} B_j$, $k \in \mathbb{N}$. It is evident that $(C_k) \subset \mathcal{F}$, is an increasing sequence and the inequality

$$p(C_k) < p(E) \tag{7}$$

holds for all $k \in \mathbb{N}$. Assume that $p(E) = p(\bigcup_{k=1}^{\infty} C_k)$. Then via (7) we obtain that $p(E) = p(E_k)$, where $E_k := \left(\bigcup_{j=1}^{\infty} C_j\right) \backslash C_k$, $k \in \mathbb{N}$. This, however, is impossible, since the sequence $(E_k) \subset \mathcal{F}$ tends decreasingly to the empty set and thus, by *Axioms 2.1* and *2.1*, $p(E_k) \to 0$, as $k \to \infty$. Hence inequality (5) holds. To end the proof, we just note that the converse is obvious. ■

Lemma 2.2.3 *For every sequence* $(B_n) \subset \mathcal{F}$ *and every optimal measure* p *we have*

$$p\left(\bigcup_{n=1}^{\infty} B_n\right) = \max\left\{p\left(B_n\right) : n \in \mathbb{N}\right\}.$$

The proof is omitted since it immediate from *Lemma 2.2.2.*

Lemma 2.2.4 *Every measurable set* $E \in \mathcal{F}$ *with* $p(E) > 0$ *contains an atom* $H \subset E$ *such that* $p(E) = p(H)$.

Proof. If E is an atom, there is nothing to be proved. We may assume that E is not an atom. Let the set $\mathcal{U} \subset \mathcal{F}$ be given:
i. if $B \in \mathcal{U}$, then $B \subset E$ and $0 < p(B) < p(E)$,
ii. if B, $C \in \mathcal{U}$ and $B \neq C$, then $B \cap C = \varnothing$.
Clearly, the collection of all such $\mathcal{U}$, denoted by $\mathcal{C}$, is partially ordered by the set inclusion. It is also obvious that every subset of $\mathcal{C}$ has an upper bound. Therefore, by the *Zorn lemma*, it follows that $\mathcal{C}$ contains a maximal element, which we shall denote by $\mathcal{U}^*$. For any fixed constant $\delta \in (0, 1)$, let us show that the set

$$\{B \in \mathcal{U}^* : p(B) > \delta\}$$

is finite. In fact, suppose that the contrary holds. Then there exists a sequence $(B_n) \subset \mathcal{U}^*$ which satisfies the inequality $p(B_n) > \delta$ for each index $n \in \mathbb{N}$. But since the sequence $E_n = \bigcup_{j=n}^{\infty} B_j$, $n \in \mathbb{N}$, tends decreasingly to the empty set, we must have that $p(E_n) \to 0$, as $n \to \infty$. This, however, contradicts the inequality $p(E_n) = \max\{p(B_j) : j = n, n+1, \ldots\} > \delta$, $n \in \mathbb{N}$. Hence $\mathcal{U}^* = \{B_k : k \in \Delta\}$ with $p(B_k) < p(E)$ for all $k \in \Delta$, where Δ is a countable index set. By *Lemma 2.2.2*, it follows that

$$p\left(\bigcup_{k\in\Delta} B_k\right) < p(E).$$

Thus it is obvious that $H = E \backslash \bigcup_{k\in\Delta} B_k$ is an atom with $p(H) = p(E)$. This completes the proof of the lemma. ■

Lemma 2.2.5 *Let* $\mathcal{H} = \{H_n : n \in J\}$ *be as above. Then for every measurable set* $B \in \mathcal{F}$ *with* $p(B) > 0$, *the identity(6.4)*

$$p\left(B \backslash \bigcup_{n\in J} (B \cap H_n)\right) = 0 \tag{8}$$

holds.

Proof. Assume that the left side of (8) were positive. Then set $B \backslash \bigcup_{n\in J} (B \cap H_n)$ would contain an atom K such that $K \cap K_n = \varnothing$ for every $K_n \in \mathcal{G}^*$. This, however, would contradict the maximality of $\mathcal{G}^*$, which ends the proof. ■

We are now in the position to prove the Structure Theorem.

Proof of the Structure Theorem. Let $\mathcal{G}$ be a set of pairwise disjoint atoms. It is clear that the collection of all such $\mathcal{G}$, denoted by Γ, is partially ordered by the set inclusion and every subset of Γ has an upper bound. Then, the *Zorn lemma* entails that Γ contains a maximal element, which we shall denote by $\mathcal{G}^*$. As we have done above, one can easily verify that the set

$$\left\{K \in \mathcal{G}^* : p(K) > n^{-1}\right\}$$

is finite. Hence $\mathcal{G}^* = \{K_j : j \in \nabla\}$, where ∇ is a countable index set. It is obvious that $p(K_j) \to 0$ as $j \to \infty$, whenever ∇ is a countably infinite set. Consequently, it ensues, via *Lemma 2.2.1*, that each atom $K_j \in \mathcal{G}^*$ can be expressed as the union of finitely many disjoint indecomposable subatoms of the same optimal measure as K_j. Finally, let us list these indecomposable atoms occurring in the decompositions of the elements of $\mathcal{G}^*$ as follows: $\mathcal{H} = \{H_n : n \in J\}$, where J is a countable index set. Now, via *Lemma 2.2.3*, the identity (8) and *Axiom 2.1*, one can easily observe that (4) holds for every set $B \in \mathcal{F}$, with $p(B) > 0$. It is also obvious that 0 is the only limit point of the set $\{p(H_n) : n \in J\}$ whenever J is a countably infinite set. This ends the proof of the theorem. ■

Definition 2.2.2 *The set $\mathcal{H}(p) = \{H_n : n \in J\}$ of disjoint indecomposable p-atoms (obtained in Theorem 2.2) will be called p-generating countable system:*
i) *it will be referred to as a p-generating infinite system and denoted by $\mathcal{H}_\infty(p)$ if J is countably infinite;*
ii) *it will be called a p-generating finite system and denoted by $\mathcal{H}_{<\infty}(p)$ if J is finite.*

To end this chapter we need to point out that, as the reader has already noticed it, we intensively made use of the *Zorn lemma* which we know is equivalent to the *axiom of choice.* In [34] an elementary proof was given to the *structure theorem.*

CHAPTER III

CHARACTERIZATION OF SOME PROPERTIES OF MEASURABLE SETS

3.1 Introduction

Some new information about σ-algebras is investigated, consisting of mapping bijectively σ-algebras onto power sets. Such σ-algebras, in fact, form a rather broad class. A special grouping of the optimal measures is used in our investigation. We constructively provide a bijective mapping that will do. In the proof we first characterize the usual set operations, the set inclusion relation as well as some asymptotic behaviors of sequences of measurable sets. Without loss of generality we shall restrict ourselves to infinite σ-algebras, since the opposite case can be easily done.

3.2 Mapping bijectively σ-algebras onto power sets

We note that this entire section is drafted from article [8].

Definition 3.2.1 (Agbeko, [8]) *We say that an optimal measure $p^* \in \mathcal{P}_\infty$ is of order-one if there is a unique indecomposable p^*-atom H such that $p^*(H) = 1$. (Any such atom will be referred to as an order-one-atom and the set of all order-one optimal measures will be denoted by $\widetilde{\mathcal{P}_\infty^1}$.)*

Example 3.2.1 *Fix a sequence $(\omega_n) \subset \Omega$ and define $p_0^* \in \mathcal{P}_\infty$ by*

$$p_0^*(B) = \max\left\{\frac{1}{n} : \omega_n \in B\right\}.$$

Then $p_0^* \in \widetilde{\mathcal{P}_\infty^1}$.

Proof. In fact, via the Structure Theorem, there is an indecomposable p_0^**-atom* H such that $p_0^*(H) = 1$. This is possible if and only if $\omega_1 \in H$. We note that there is no other indecomposable p_0^*-atom H^* with $H^* \cap H = \emptyset$ such that $p_0^*(H^*) = 1$, otherwise necessarily it would ensue that $\omega_1 \in H^*$, which is absurd. Therefore, we can conclude that $p_0^* \in \widetilde{\mathcal{P}_\infty^1}$. ■

Further notations

If H is the order-one-atom of some $p^* \in \widetilde{\mathcal{P}_\infty^1}$, we write $p = \left\{q^* \in \widetilde{\mathcal{P}_\infty^1} : q^*(H) = 1\right\}$. We then refer to the elements of the class p as representing members of the class, and call H the *unitary atom of the class.*

We further denote by $\mathcal{P}_\infty^1$ the set of all p classes.

If A is a nonempty measurable set and $p \in \mathcal{P}_\infty^1$, the identity $p(A) = 1$ (resp. the inequality $p(A) < 1$) will simply mean that $p^*(A) = 1$ (resp. $p^*(A) < 1$) for any representing member $p^* \in p$. We shall also write $p(A) = 0$ to mean that $p^*(A) = 0$ whenever $p^* \in p$.

Write ∇ for the set of all unitary atoms on the measurable space $(\Omega, \mathcal{F})$.

Lemma 3.2.1 *Let $A, B \in \mathcal{F}$ and $p \in \mathcal{P}_\infty^1$ be arbitrary. In order that $p(A \cap B) = 1$ it is necessary and sufficient that $p(A) = 1$ and $p(B) = 1$.*

Proof. As the necessity is obvious, we only need show the sufficiency. In fact, assume that $p(A) = 1$ and $p(B) = 1$. Let H be the unitary atom of class p, and let p^* denote an arbitrary but fixed representing member in the class. Thus H is an order-one-atom for p^*. Then $p^*(H) = 1$. Clearly, $p^*(A \cap H) = 1$ and $p^*(B \cap H) = 1$. Hence $p^*\left(\overline{A} \cap H \cap \overline{B}\right) = 0$. It is enough to prove that both identities $p^*\left(A \cap H \cap \overline{B}\right) = 0$ and $p^*\left(\overline{A} \cap H \cap B\right) = 0$ are valid. In the contrary, assume that at least one of these identities fails to hold: $p^*\left(A \cap H \cap \overline{B}\right) = 0$, say. Then $p^*\left(A \cap H \cap \overline{B}\right) = 1$. Now, since $p^*(H \cap B) = 1$, it ensues that either $p^*(A \cap H \cap B) = 1$ or $p^*\left(\overline{A} \cap H \cap B\right) = 1$. Then combining each of these last identities with $p^*\left(A \cap H \cap \overline{B}\right) = 1$, we have that $p^*\left(A \cap H \cap \overline{B}\right) = 1$ and $p^*(A \cap H \cap B) = 1$, or $p^*\left(A \cap H \cap \overline{B}\right) = 1$ and $p^*\left(\overline{A} \cap H \cap B\right) = 1$. This violates that H is an order-one-atom (because the sets $A \cap H \cap B$, $A \cap H \cap \overline{B}$ and $\overline{A} \cap H \cap B$ are pairwise disjoint). ■

Remark 3.2.1 *Let $p \in \mathcal{P}_\infty^1$ be arbitrary. Then the identity $p(\emptyset) = 0$ holds (cf. Axiom 2.1).*

Remark 3.2.2 *Let $A \in \mathcal{F}$ and $p \in \mathcal{P}_\infty^1$ be arbitrary. Then the identities $p(A) = 1$ and $p\left(\overline{A}\right) = 1$ cannot hold simultaneously, i.e. for no representing member p^* of class p the identities $p^*(A) = 1$ and $p^*\left(\overline{A}\right) = 1$ hold at the same time.*

In fact, assume the contrary. Then *Lemma 3.2.1* would imply that

$$1 = p(A) = p\left(\overline{A}\right) = p\left(A \cap \overline{A}\right) = p(\emptyset) = 0$$

which is absurd, indeed.

Definition 3.2.2 (Agbeko, [8]) *For any $A \in \mathcal{F}$ let the set $\Delta(A)$ be described by*

1. $\Delta(A) \subseteq \mathcal{P}_\infty^1$.
2. If $p \in \Delta(A)$, then $p(A) = 1$.

Remark 3.2.3 *Let $A \in \mathcal{F}$. Then $\Delta(A) = \emptyset$ if and only if $A = \emptyset$.*

Remark 3.2.4 *If H is the unitary atom of a class $p \in \mathcal{P}_\infty^1$, then $\Delta(H) = \{p\}$.*

Let $A \in \mathcal{F}$ and denote by ∇_A the set of all unitary atoms H such that $p(A) = 1$, where $\Delta(H) = \{p\}$. It is clear that $\nabla_A \cap \nabla_{\overline{A}} = \emptyset$ and $\nabla_A \cup \nabla_{\overline{A}} = \nabla$. From this observation the following lemma is straightforward.

Lemma 3.2.2 *For every set $A \in \mathcal{F}$, we have that $\Delta\left(\overline{A}\right) = \overline{\Delta(A)}$.*

Proposition 3.2.1 *Let $A, B \in \mathcal{F}$ be arbitrary. Then*

1. $\Delta(\Omega) = \mathcal{P}_{\infty}^{1}$.
2. $\Delta(A \cap B) = \Delta(A) \cap \Delta(B)$.
3. $\Delta(A \cup B) = \Delta(A) \cup \Delta(B)$.

Proof. *Part 1* is an easy task. Let us show *Part 2*. In fact, let $p \in \Delta(A \cap B)$. Then $p(A \cap B) = 1$. Hence *Lemma 3.2.1* implies that $p(A) = 1$ and $p(B) = 1$, so that $p \in \Delta(A)$ and $p \in \Delta(B)$, i.e. $p \in \Delta(A) \cap \Delta(B)$. Consequently, $\Delta(A \cap B) \subseteq \Delta(A) \cap \Delta(B)$. To show the reverse inclusion, pick an arbitrary $p \in \Delta(A) \cap \Delta(B)$. Then $p(A) = 1$ and $p(B) = 1$. Via *Lemma 3.2.1*, we have that $p(A \cap B) = 1$, i.e. $p \in \Delta(A \cap B)$. So $\Delta(A) \cap \Delta(B) \subseteq \Delta(A \cap B)$.

To end the proof, let us show the third part. In fact, let A and $B \in \mathcal{F}$ be arbitrary. Then making use of the second part of this proposition it ensues that $\Delta\left(\overline{A} \cap \overline{B}\right) = \Delta\left(\overline{A}\right) \cap \Delta\left(\overline{B}\right)$. By applying *Lemma 3.2.2* and the De Morgan identities, we obtain that

$$\begin{aligned} \Delta(A \cup B) &= \overline{\overline{\Delta(A \cup B)}} = \overline{\Delta\left(\overline{A} \cap \overline{B}\right)} = \overline{\Delta\left(\overline{A}\right) \cap \Delta\left(\overline{B}\right)} \\ &= \overline{\Delta\left(\overline{A}\right)} \cup \overline{\Delta\left(\overline{B}\right)} = \overline{\overline{\Delta(A)}} \cup \overline{\overline{\Delta(B)}} = \Delta(A) \cup \Delta(B). \end{aligned}$$

This was to be proven. ■

Lemma 3.2.3 *Let A and $B \in \mathcal{F}$ be arbitrary nonempty sets. In order that $A \subset B$, it is necessary and sufficient that $\Delta(A) \subset \Delta(B)$.*

Proof. As the necessity is trivial we need only show the sufficiency. In fact, assume that $A \backslash B$ is not an empty set. Then because of *Remark 3.2.3*, $\Delta(A \backslash B)$ is neither empty. Fix some $p \in \Delta(A \backslash B)$, i.e. $p(A \backslash B) = 1$. This implies that $p(B) < 1$. Otherwise we would obtain via *Lemma 3.2.1* that $1 = p((A \backslash B) \cap B) = p(\emptyset) = 0$, which is absurd. Then $p(A) = 1$ and $p(B) < 1$, i.e. $p \in \Delta(A) \backslash \Delta(B)$. So the set $\Delta(A) \backslash \Delta(B)$ is not empty. ■

Lemma 3.2.4 *Let A and $B \in \mathcal{F}$ be arbitrary nonempty sets. Then for the equality $A \cap B = \emptyset$ to hold it is necessary and sufficient that $\Delta(A) \cap \Delta(B) = \emptyset$.*

(The proof follows from *Proposition 3.2.1/2* and *Remark 3.2.3*.)

Lemma 3.2.5 *Let A and $B \in \mathcal{F}$ be arbitrary nonempty sets. In order that $A = B$ it is necessary and sufficient that $\Delta(A) = \Delta(B)$.*

Proof. As the necessity is trivial we need only show the sufficiency. In fact, assume that A and $B \in \mathcal{F}$ are such that $\Delta(A) = \Delta(B)$, i.e. $\Delta(A) \subseteq \Delta(B)$ and $\Delta(B) \subseteq \Delta(A)$. By applying twice *Lemma 3.2.3* it ensues that $A \subseteq B$ and $B \subseteq A$. Therefore, $A = B$. ■

Lemma 3.2.6 *Let A and $B \in \mathcal{F}$ be any nonempty sets. Then $\Delta(A \backslash B) = \Delta(A) \backslash \Delta(B)$.*

Proof. The conjunction of *Proposition 3.2.1/2* and *Lemma 3.2.2* entails that

$$\begin{aligned}\Delta(A \backslash B) &= \Delta\left(A \cap \overline{B}\right) = \Delta(A) \cap \Delta\left(\overline{B}\right) \\ &= \Delta(A) \cap \left(\overline{\Delta(B)}\right) = \Delta(A) \backslash \Delta(B),\end{aligned}$$

which completes the proof. ■

Proposition 3.2.2 *Let $(A_n) \subset \mathcal{F}$ and $A \in \mathcal{F}$ be arbitrary. Then (A_n) converges decreasingly to A if, and only if $(\Delta(A_n))$ converges decreasingly to $\Delta(A)$.*

Proof. Assume that (A_n) converges decreasingly to A. Then by applying repeatedly *Lemma 3.2.3* we have for every $n \in \mathbb{N}$ that

$$\Delta(A) \subset \Delta(A_{n+1}) \subset \Delta(A_n).$$

We need to prove that $\Delta(A) = \bigcap_{n=1}^{\infty} \Delta(A_n)$. To do this it will be enough to show that $\Delta(A) \subseteq \bigcap_{n=1}^{\infty} \Delta(A_n)$ and $\bigcap_{n=1}^{\infty} \Delta(A_n) \subseteq \Delta(A)$. In fact, we note that the first inclusion is trivial. To prove the second inclusion let us pick some $p \in \bigcap_{n=1}^{\infty} \Delta(A_n)$. Then $p \in \Delta(A_n)$ for all $n \in \mathbb{N}$. Hence $p(A_n) = 1$ for all $n \in \mathbb{N}$. If we fix any representing member p^* in class p we then obtain via *Axiom 2.1* that

$$p^*(A) = p^*\left(\bigcap_{n=1}^{\infty} A_n\right) = \min\{p^*(A_n) : n \in \mathbb{N}\} = 1,$$

implying that $p(A) = 1$, i.e. $p \in \Delta(A)$. Consequently, $\bigcap_{n=1}^{\infty} \Delta(A_n) \subseteq \Delta(A)$.

Conversely, assume that sequence $(\Delta(A_n))$ converges decreasingly to $\Delta(A)$. Then for every $n \in \mathbb{N}$ we obtain that $\Delta(A) \subset \Delta(A_{n+1}) \subset \Delta(A_n)$ so that $A \subset A_{n+1} \subset A_n$, $n \in \mathbb{N}$ (by *Lemma 3.2.3*). Hence $A \subseteq \bigcap_{n=1}^{\infty} A_n$. To show the reverse inclusion let us assume that set $\left(\bigcap_{n=1}^{\infty} A_n\right) \backslash A$ is not empty. Then via *Remark 3.2.3* and *Axiom 2.1* there can be found some $p \in \mathcal{P}_{\infty}^1$ such that

$$1 = p^*\left(\left(\bigcap_{n=1}^{\infty} A_n\right) \backslash A\right) = p^*\left(\bigcap_{n=1}^{\infty} A_n \cap \overline{A}\right) = \min\left\{p^*\left(A_n \cap \overline{A}\right) : n \in \mathbb{N}\right\},$$

for every representing member p^* of class p, since (A_n) is a decreasing sequence. Consequently, $1 = p^*\left(A_n \cap \overline{A}\right)$ for all $n \in \mathbb{N}$. Hence *Lemma 3.2.2* yields that $p\left(\overline{A}\right) = 1$ and $p(A_n) = 1$ for all $n \in \mathbb{N}$. But then $p \in \Delta(A_n)$ for all $n \in \mathbb{N}$ and hence $p \in \bigcap_{n=1}^{\infty} \Delta(A_n) = \Delta(A)$. Nevertheless, this is absurd since $p \in \Delta\left(\overline{A}\right) = \overline{\Delta(A)}$. We can thus conclude on the validity of the proposition. ■

Proposition 3.2.3 *Let $(A_n) \subset \mathcal{F}$ and $A \in \mathcal{F}$ be arbitrary. Then (A_n) converges increasingly to A if and only if $(\Delta(A_n))$ converges increasingly to $\Delta(A)$.*

Proof. Assume that (A_n) converges increasingly to A. Then by applying repeatedly *Lemma 3.2.3* we have for every $n \in \mathbb{N}$ that

$$\Delta(A_n) \subset \Delta(A_{n+1}) \subset \Delta(A).$$

We need to prove that $\Delta(A) = \bigcup_{n=1}^{\infty} \Delta(A_n)$. To do this it will be enough to show that $\Delta(A) \subseteq \bigcup_{n=1}^{\infty} \Delta(A_n)$ and $\bigcup_{n=1}^{\infty} \Delta(A_n) \subseteq \Delta(A)$. In fact, we note that the second inclusion is trivial. To prove the first one let us pick an arbitrary class $p \in \Delta(A)$ and fix any representing member p^* of the class p. Following the proof of *Lemma 0.1* (cf. [5], page 134), there can be found a positive integer n_0 such that $1 = p^*(A) = p^*\left(\bigcup_{k=1}^{\infty} A_k\right) = p^*(A_n)$, whenever $n \geq n_0$. Hence $p \in \bigcup_{n=n_0}^{\infty} \Delta(A_n) \subseteq \bigcup_{n=1}^{\infty} \Delta(A_n)$, i.e.

$$\Delta(A) \subseteq \bigcup_{n=n_0}^{\infty} \Delta(A_n) \subseteq \bigcup_{n=1}^{\infty} \Delta(A_n).$$

Conversely, assume that sequence $(\Delta(A_n))$ converges increasingly to $\Delta(A)$. Then sequence $\left(\overline{\Delta(A_n)}\right)$ converges decreasingly to $\overline{\Delta(A)}$. Consequently, *Lemma 3.2.2* entails that sequence $\left(\Delta\left(\overline{A_n}\right)\right)$ converges decreasingly to $\Delta\left(\overline{A}\right)$. Taking into account *Proposition 3.2.2*, sequence $\left(\overline{A_n}\right)$ must converge decreasingly to $\overline{A}$. In turn this implies that (A_n) converges increasingly to A.

Therefore, we can conclude on the validity of the argument. ■

Theorem 3.2.1 (Agbeko, [8]) *Let $(A_n) \subset \mathcal{F}$ and $A \in \mathcal{F}$ be arbitrary. In order that (A_n) converge to A it is necessary and sufficient that $(\Delta(A_n))$ converge to $\Delta(A)$.*

Proof. For every counting number $n \in \mathbb{N}$ write $E_n = \bigcap_{k=n}^{\infty} A_k$ and $B_n = \bigcup_{k=n}^{\infty} A_k$. It is clear that sequence (B_n) converges decreasingly to $\limsup_{n\to\infty} A_n$ and sequence (E_n) converges increasingly to $\liminf_{n\to\infty} A_n$. Consequently, by applying *Propositions 3.2.2* and *3.2.3* to these sequences we can conclude on the validity of the theorem. ■

Definition 3.2.3 (Agbeko, [8]) *A mapping $\Delta : \mathcal{F} \to \mathbb{P}(\mathcal{P}_\infty^1)$ is said to be* powering *if it is defined by:*

$$\Delta(A) = \begin{cases} \emptyset & \text{if } A = \emptyset \\ \{p \in \mathcal{P}_\infty^1 : p(A) = 1\} & \text{if } A \neq \emptyset \end{cases}$$

The following result can easily be derived from *Lemma 3.2.5* and *Remark 3.2.3*.

Proposition 3.2.4 *If $\Delta : \mathcal{F} \to \mathbb{P}(\mathcal{P}_\infty^1)$ is a powering mapping, then it is an injection.*

Definition 3.2.4 *If $\Gamma \subseteq \mathcal{P}_{\infty}^{1}$ is a nonempty set, then the collection $\mathcal{C}$ of all the unitary atoms of the classes $p \in \Gamma$ will be called unitary-atomic (or governing-atomic) collection of Γ.*

The Postulate of Powering. *If* $\Gamma \in \mathbb{P}(\mathcal{P}_{\infty}^{1}) \setminus \{\emptyset\}$ and $\mathcal{C}$ *denotes the governing-atomic collection of Γ, then $\bigcup \mathcal{C}$ is measurable and $\Delta(\bigcup \mathcal{C}) \subseteq \Gamma$.*

Theorem 3.2.2 (Agbeko, [8]) *The powering mapping $\Delta : \mathcal{F} \to \mathbb{P}(\mathcal{P}_{\infty}^{1})$ is surjective if and only if the postulate of powering is valid.*

Proof. Assume that the *postulate of powering* is valid. Let $\Gamma \in \mathbb{P}(\mathcal{P}_{\infty}^{1})$ be arbitrarily fixed. We note that if $\Gamma = \emptyset$, then there is nothing to be proven. Suppose that Γ is a nonempty subset of $\mathcal{P}_{\infty}^{1}$, and denote by $\mathcal{C}$ its corresponding governing-atomic collection. Then $\bigcup \mathcal{C}$ is measurable and $\Delta(\bigcup \mathcal{C}) \subseteq \Gamma$ (by the postulate). Let us show that $\Gamma \subseteq \Delta(\bigcup \mathcal{C})$. In fact, pick any class $p \in \Gamma$ and p^* any representing member of p, with H the unitary atom of p. Since $H \subseteq \bigcup \mathcal{C}$, it ensues from *Lemma 3.2.2* that $\Delta(H) \subseteq \Delta(\bigcup \mathcal{C})$. But, via *Remark 3.2.4* we have that $\{p\} = \Delta(H)$ and thus $p \in \Delta(\bigcup \mathcal{C})$, i.e. $\Gamma \subseteq \Delta(\bigcup \mathcal{C})$. Therefore, $\Gamma = \Delta(\bigcup \mathcal{C})$.

To prove the converse of the biconditional, let us assume that the powering mapping Δ is a surjection. We note consequently that Δ is a bijection, since it is also an injection (by *Proposition 3.2.4*). Let $\Gamma \in \mathbb{P}(\mathcal{P}_{\infty}^{1}) \setminus \{\emptyset\}$ be arbitrary and write $\mathcal{C}$ for the corresponding unitary-atomic collection. Obviously, we have that $\Gamma = \bigcup \{\Delta(H) : H \in \mathcal{C}\}$ is a subset of $\mathcal{P}_{\infty}^{1}$. Then via the bijective property it ensues that $\Delta^{-1}(\Gamma) \in \mathcal{F}$. Clearly, $\Delta(H) \subset \Gamma$ for every $H \in \mathcal{C}$. By *Lemma 3.2.3* together with the bijective property, we obtain that

$$H = \Delta^{-1}(\Delta(H)) \subset \Delta^{-1}(\Gamma)$$

whenever $H \in \mathcal{C}$. Consequently, the inclusion $\bigcup \mathcal{C} \subseteq \Delta^{-1}(\Gamma)$ follows. Now, let us show that if $\omega \in \Delta^{-1}(\Gamma)$, then there is some $H \in \mathcal{C}$ such that $\omega \in H$. Assume in the contrary that there can be found some $\omega_1 \in \Delta^{-1}(\Gamma)$ such that $\omega_1 \notin H$ for all $H \in \mathcal{C}$. We can thus define an optimal measure $q^* : \mathcal{F} \to [0,\, 1]$ so that

$$q^*(B) \begin{cases} = & 1 \quad \text{if } \omega_1 \in B \\ < & 1 \quad \text{if } \omega_1 \notin B, \end{cases}$$

see *Example 3.2.1*. Then there is a unique indecomposable q^*-atom (to be denoted by $\widetilde{H}$) such that $q^*\left(\widetilde{H}\right) = 1$. Obviously, $\omega_1 \in \widetilde{H}$ and $q^*(\Delta^{-1}(\Gamma)) = 1$. We further note that

$$\bigcup \{\Delta(H) : H \in \mathcal{C}\} = \Gamma = \Delta\left(\Delta^{-1}(\Gamma)\right) = \left\{p \in \mathcal{P}_{\infty}^{1} : p\left(\Delta^{-1}(\Gamma)\right) = 1\right\}.$$

From this fact and the identity $q^*(\Delta^{-1}(\Gamma)) = 1$, there must exist some class $p_0 \in \mathcal{P}_{\infty}^{1}$ with $p_0(\Delta^{-1}(\Gamma)) = 1$, such that $q^*\left(\widetilde{H} \cap H_0 \cap \Delta^{-1}(\Gamma)\right) = 1$, where $H_0 \in \mathcal{C}$ is the unitary atom of class p_0. Nevertheless, this is possible only if $\omega_1 \in H_0$, which is absurd, since earlier we have supposed that $\omega_1 \notin H$ for all $H \in \mathcal{C}$. Therefore, if $\omega \in \Delta^{-1}(\Gamma)$, then there is some $H \in \mathcal{C}$ such that $\omega \in H$. It ensues that $\omega \in \bigcup \mathcal{C}$ for all $\omega \in \Delta^{-1}(\Gamma)$, as $H \subset \bigcup \mathcal{C}$

whenever $H \in \mathcal{C}$. Thus $\Delta^{-1}(\Gamma) \subseteq \bigcup \mathcal{C}$. Therefore, $\bigcup \mathcal{C} = \Delta^{-1}(\Gamma)$, which leads to the postulate. ■

Theorem 3.2.2 entails that an infinite σ-algebra is equinumerous with a power set if and only if *Postulate 3.2* is valid. This suggests that every infinite σ-algebra is either equinumerous with an infinite power set or with a non-power set.

CHAPTER IV

SOME BASIC RESULTS OF OPTIMAL MEASURES RELATED TO MEASURABLE FUNCTIONS

4.1 Introduction

In comparison with the mathematical expectation, we shall define a non-linear functional (first for non-negative measurable simple functions and secondly for non-negative measurable functions) which can provide us with many well-known results in measure theory. Their proofs are carried out similarly.

4.2 Optimal average

In the whole section we shall be dealing with an arbitrary but fixed optimal measure space $(\Omega, \mathcal{F}, p)$.

Let

$$s = \sum_{i=1}^{n} b_i \chi(B_i)$$

be an arbitrary non-negative measurable simple function, where $\{B_i : i = 1, \ldots, n\} \subset \mathcal{F}$ is a partition of Ω. Then the so-called optimal average of s is defined by

Definition 4.2.1 *The quantity*

$$\Big\backslash_{\Omega} s dp := \bigvee_{i=1}^{n} b_i p(B_i)$$

will be called optimal average of s, and for $E \in \mathcal{F}$

$$\Big\backslash_{B} s\chi(E)\, dp := \bigvee_{i=1}^{n} b_i p(E \cap B_i)$$

as the optimal average of s on E, where $\chi(E)$ is the indicator function of the measurable set E. These quantities will be sometimes denoted respectively by $I(s)$ and $I_E(s)$.

It is well-known that in general a measurable simple function has many decompositions. The question thus arises whether or not the optimal average depends on the decomposition of the simple function. The following result gives a satisfactory answer to this question.

Theorem 4.2.1 (Agbeko, [5]) *Let*

$$\sum_{i=1}^{n} b_i \chi(B_i) \quad \textit{and} \quad \sum_{k=1}^{m} c_k \chi(C_k)$$

be two decompositions of a measurable simple function $s \geq 0$, where $\{B_i : i = 1, \ldots, n\}$ and $\{C_k : k = 1, \ldots, m\} \subset \mathcal{F}$ are partitions of Ω. Then

$$\max\{b_i p(B_i) : i = 1, \ldots, n\} = \max\{c_k p(C_k) : k = 1, \ldots, m\}.$$

Proof. Since $B_i = \bigcup_{k=1}^{m} (B_i \cap C_k)$ and $C_k = \bigcup_{i=1}^{n} (B_i \cap C_k)$, *Axiom 2.1* of optimal measure implies that

$$p(B_i) = \max\{p(B_i \cap C_k) : k = 1, \ldots, m\} \text{ and } p(C_k) = \max\{p(B_i \cap C_k) : i = 1, \ldots, n\}$$

Thus

$$\max\{c_k p(C_k) : k = 1, \ldots, m\} = \max\{\max\{c_k p(B_i \cap C_k) : i = 1, \ldots, n\} : k = 1, \ldots, m\}$$

and

$$\max\{b_i p(B_i) : i = 1, \ldots, n\} = \max\{\max\{b_i p(B_i \cap C_k) : k = 1, \ldots, m\} : i = 1, \ldots, n\}.$$

Clearly, if $B_i \cap C_k \neq \varnothing$, then $b_i = c_k$, or if $B_i \cap C_k = \varnothing$, then $p(B_i \cap C_k) = 0$. Thus, by the associativity and the commutativity, we obtain

$$\max\{b_i p(B_i) : i = 1, \ldots, n\} = \max\{c_k p(C_k) : k = 1, \ldots, m\}.$$

This completes the proof. ■

Theorem 4.2.2 *Let s and $\overline{s}$ denote two non-negative measurable simple functions, $b \in [0, \infty]$ and $B \in \mathcal{F}$ be arbitrary. Then we have:*

1. $I(b\mathbf{1}) = b$.
2. $I(\chi(B)) = p(B)$.
3. $I(bs) = bI(s)$.
4. $I_B(s) = 0$ *if* $p(B) = 0$.
5. $I(s) = I_B(s)$ *if* $p\left(\overline{B}\right) = 0$.
6. $I(s) \leq I(\overline{s})$ *if* $s \leq \overline{s}$ *on* Ω.
7. $I(s + \overline{s}) \leq I(s) + I(\overline{s})$.
8. $I_B(s) = \lim_{n\to\infty} I_{B_n}(s)$ *whenever* $(B_n) \subset \mathcal{F}$ *tends increasingly to* B.

9. $I(s \vee \overline{s}) = \max\{I(s), I(\overline{s})\}$.

The proof is omitted because it is based on computation only.

Proposition 4.2.1 *Let $f \geq 0$ be any bounded measurable function. Then*

$$\sup_{s \leq f} \diagdown_{\Omega} s\,dp = \inf_{\overline{s} \geq f} \diagdown_{\Omega} \overline{s}\,dp,$$

where s and $\overline{s}$ denote non-negative measurable simple functions.

Proof. Let f be a measurable function such that $0 \leq f \leq b$ on Ω, where b is some constant. Let $E_k = (kbn^{-1} \leq f \leq (k+1)bn^{-1})$, $k = 1, \ldots, n$. Clearly, $\{E_k : k = 1, \ldots, n\} \subset \mathcal{F}$ is a partition of Ω. Define the following measurable simple functions:

$$s_n = bn^{-1} \sum_{k=0}^{n} k\chi(E_k), \quad \overline{s}_n = bn^{-1} \sum_{k=0}^{n} (k+1)\chi(E_k).$$

Obviously, $s_n \leq f \leq \overline{s}_n$. Then we can easily observe that

$$\sup_{s \leq f} \diagdown_{\Omega} s\,dp \geq \diagdown_{\Omega} s_n\,dp = n^{-1}b \max\{kp(E_k) : k = 0, \ldots, n\}$$

and

$$\inf_{\overline{s} \geq f} \diagdown_{\Omega} \overline{s}\,dp \leq \diagdown_{\Omega} \overline{s}_n\,dp = n^{-1}b \max\{(k+1)p(E_k) : k = 0, \ldots, n\}.$$

Hence

$$0 \leq \inf_{\overline{s} \geq f} \diagdown_{\Omega} \overline{s}\,dp - \sup_{s \leq f} \diagdown_{\Omega} s\,dp \leq bn^{-1}.$$

The result follows by letting $n \to \infty$ in this last inequality. ■

Definition 4.2.2 (Agbeko, [5]) *The optimal average of a measurable function f is defined by $\diagdown_{\Omega} |f|\,dp = \sup \diagdown_{\Omega} s\,dp$, where the supremum is taken over all measurable simple functions $s \geq 0$ for which $s \leq |f|$. The optimal average of f on any given measurable set E is defined by $\diagdown_{E} |f|\,dp = \diagdown_{\Omega} \chi(E)|f|\,dp$.*

For convenience reasons at times we shall write $A|f|$ for the optimal average of the measurable function f.

Proposition 4.2.2 (Agbeko, [5]) *Let $f \geq 0$ and $g \geq 0$ be any measurable simple functions, $b \in \mathbb{R}_+$ and $B \in \mathcal{F}$ be arbitrary. Then*

1. $A(b\mathbf{1}) = b$.

2. $A(\chi(B)) = p(B)$.

3. $A(bf) = bAf$.

4. $A(f\chi(B)) = 0$ *if* $p(B) = 0$.

5. $Af \leq Ag$ *if* $f \leq g$.

6. $A(f+g) \leq Af + Ag$.

7. $A(f\chi(B)) = Af$ *if* $p\left(\overline{B}\right) = 0$.

8. $A(\max\{f, g\}) = \max\{Af, Ag\}$.

The almost everywhere notion in measure theory also makes sense in optimal measure theory.

Definition 4.2.3 *Let p be an optimal measure. A property is said to hold almost everywhere if the set of elements where it fails to hold is a set of optimal measure zero.*

As an immediate consequent of the atomic structural behavior of optimal measures we can formulate the following.

Remark 4.2.1 (Agbeko, [6]) *If a function $f : \Omega \to \mathbb{R}$ is measurable, then it is constant almost everywhere on every indecomposable atom.*

Proposition 4.2.3 (Agbeko, [6]) *Let $p \in \mathcal{P}$ and f be any measurable function. Then*

$$\bigvee_{\Omega} |f| \, dp = \sup \left\{ \bigvee_{H_n} |f| \, dp : n \in J \right\},$$

where $\mathcal{H}(p) = \{H_n : n \in J\}$ is a p-generating countable system.
Moreover if $A|f| < \infty$, then $\bigvee_{\Omega} |f| \, dp = \sup\{c_n \cdot p(H_n) : n \in J\}$, *where $c_n = f(\omega)$ for almost all $\omega \in H_n$, $n \in J$.*

Proposition 4.2.4 (Optimal Markov inequality, [5]) *Let $f \geq 0$ be any measurable function. Then for every number $x > 0$ we have*

$$xp(f \geq x) \leq Af.$$

Proposition 4.2.5 *Let $f \geq 0$ be any measurable function and $b > 0$ be any number.*

1. *If $Af < \infty$, then $f < \infty$ almost everywhere.*

2. *$Af = 0$ if and only if $f = 0$ almost everywhere.*

3. *If $Af < \infty$, then $f < \infty$ almost everywhere.*

4. *If* $Af < \infty$ *and* $\frac{1}{p(E)} \int_E f dp \geq b$ *for all* $E \in \mathcal{F}$ *with* $p(E) > 0$*, then* $f \geq b$ *almost everywhere*

5. *If* $Af < \infty$ *and* $\frac{1}{p(E)} \int_E f dp \leq b$ *for all* $E \in \mathcal{F}$ *with* $p(E) > 0$*, then* $f \leq b$ *almost everywhere.*

Proposition 4.2.6 *Let* $f \geq 0$ *be any bounded measurable function. Then for every* $\varepsilon > 0$ *there is some* $\delta > 0$ *such that* $\int_B f dp < \varepsilon$ *whenever* $B \in \mathcal{F}$, $p(B) < \delta$.

Proof. By assumption $0 \leq f \leq b$ for some number $b > 0$. Then *Proposition 4.2.2* entails, for the choice $0 < \delta < \varepsilon b^{-1}$, that $\int_B f dp \leq bp(B) < \delta b < \varepsilon$. ■

In the example below we shall show that *Proposition 4.2.6* does not hold for unbounded measurable functions.

Example 4.2.1 *Consider the measurable space* $(\mathbb{N}, \mathcal{F})$*, where* $\mathcal{F}$ *is the power set of* $\mathbb{N}$*. Define the set function* $p : \mathcal{F} \to [0, 1]$ *by* $p(B) = \dfrac{1}{\min B}$*. It is not difficult to see that* p *is an optimal measure. Consider the following measurable function* $f(\omega) = \omega$, $\omega \in \mathbb{N}$*. Clearly,* $Af \geq 1$*. Let* $s = \sum_{j=1}^{n} b_j \chi(B_j)$ *be a measurable simple function with* $0 \leq s \leq f$*. Denote* $\omega_j = \min B_j$ *for* $j = 1, \ldots, n$*. Then* $p(B_j) = \dfrac{1}{\omega_j}$ *and* $b_j \leq \omega_j$ *for all* $j = 1, \ldots, n$*. Thus* $I(s) \leq 1$*, and hence* $Af \leq 1$*. Consequently,* $Af = 1$*. On the one hand, there is no* $\delta > 0$ *such that* $p(E) < \delta$ *implies that* $\int_E f dp < 1$*. Indeed,* $\int_{\{\omega\}} f dp = 1$ *for every* $\omega \in \mathbb{N}$*, and* $p(\{\omega\}) \to 0$ *as* $\omega \to \infty$*.*

4.3 The Radon-Nikodym's type theorem

Definition 4.3.1 (Agbeko, [6]) *By a quasi-optimal measure we a set function* $q : \mathcal{F} \to \mathbb{R}_+$ *satisfying Axioms 2.1-2.1, with the hypothesis* $q(\Omega) = 1$ *in Axiom 2.1 being replaced by the hypothesis* $0 < q(\Omega) < \infty$*.*

Proposition 4.3.1 *If* $f \geq 0$ *is a bounded measurable function, then the set function* $q_f : \mathcal{F} \to \mathbb{R}_+$*,*

$$q_f(E) = \int_E f dp,$$

is a quasi-optimal measure.

Definition 4.3.2 *We shall say that a quasi-optimal measure* q *is absolutely continuous relative to* p *(abbreviated* $q \ll p$*) if* $q(B) = 0$ *whenever* $p(B) = 0$, $B \in \mathcal{F}$*.*

Proposition 4.3.2 *Let* q *be a quasi-optimal measure. Then* $q \ll p$ *if and only if for every* $\varepsilon > 0$ *there is some* $\delta > 0$ *such that* $q(B) < \varepsilon$ *whenever* $p(B) < \delta$, $B \in \mathcal{F}$*.*

The proof of *Proposition 4.3.2* is similarly done as in the case of measure theory.

Lemma 4.3.1 *Let q be a quasi-optimal measure and $\mathcal{H}(p)$ be a p-generating system. If $q \ll p$, then*

$$\mathcal{H}(q) = \{H \in \mathcal{H}(p) : q(H) > 0\}$$

is a q-generating system.

Proof. Let H be an indecomposable p-atom. Suppose that there exists a measurable set $E \subset H$ with $q(E) = q(H \backslash E) = q(H) > 0$. Since $q \ll p$, it must ensue that $p(E) > 0$ and $p(H \backslash E) > 0$, contradicting the fact that H is an indecomposable p-atom. Hence we can conclude that every indecomposable p-atom is also H be an indecomposable q-atom whenever $q(H) > 0$ and observe that

$$\mathcal{H}(q) = \{H \in \mathcal{H}(p) : q(H) > 0\} = \{H_k \in \mathcal{H}(p) : k \in J^*\},$$

where $J^* \subseteq J$ is an index set.

Let B be any measurable set with $q(B) > 0$. Then, via *Lemma 2.2.5* and the absolute continuity property it follows that

$$q\left(B \backslash \bigcup_{k \in J^*} (B \cap H_k)\right) = 0.$$

Thus $q(B) = \max\{q(B \cap H_k) : k \in J^*\}$.

If J^* is a countably infinite set, then *Proposition 4.3.2* yields that $q(H_k)$ becomes arbitrarily small along with $p(H_k)$ as $k \to \infty$. This ends the proof. ■

Remark 4.3.1 *Let $p, q \in \mathcal{P}, \mathcal{H}(p) = \{H_n : n \in J\}$ be a p-generating countable system and f any measurable function. Suppose that $q \ll p$ and $q(H) \leq p(H)$ for every $H \in \mathcal{H}(p)$. Then $\int_\Omega |f| \, dq \leq \int_\Omega |f| \, dp$, provided that $\int_\Omega |f| \, dp < \infty$.*

This remark is immediate from *Lemma 4.3.1* and *Proposition 4.2.3.*

Theorem 4.3.1 (Optimal Radon-Nikodym, [6]) *Let q be a quasi-optimal measure such that $q \ll p$. Then there exists a unique measurable function $f \geq 0$ such that for every measurable set $B \in \mathcal{F}$,*

$$q(B) = \int_B f dp.$$

This measurable function, explicitly given in (9), will be called Optimal Radon-Nikodym derivative and denoted by $\dfrac{dq}{dp}$.

Proof. Let $\mathcal{H}(p) = \{H_n : n \in J\}$ be a p-generating countable system. Define the following non-negative measurable function

$$f = \max\left\{\frac{q(H_n)}{p(H_n)} \cdot \chi(H_n) : n \in J\right\}. \tag{9}$$

Fix an index $n \in J$ and let $B \in \mathcal{F}$, $p(B) > 0$. Then *Remark 2.2.1* and the absolute continuity property imply that

$$\frac{q(H_n)}{p(H_n)} p(B \cap H_n) = \begin{cases} 0 & \text{if } p(B \cap H_n) = 0 \\ q(B \cap H_n), & \text{otherwise.} \end{cases}$$

Hence, by a simple calculation, one can observe that

$$\int_B f dp = \max\{q(B \cap H_n) : n \in J\}.$$

Consequently, *Lemma 4.3.1* yields

$$\int_B f dp = \begin{cases} \max\{q(B \cap H_n) : q(H_n) > 0,\, n \in J\} & \text{if } q(B) > 0 \\ 0, & \text{otherwise,} \end{cases}$$

and thus (9) holds.

Let us show that the decomposition (9) is unique. In fact, there exist two measurable functions $f \geq 0$ and $g \geq 0$ satisfying(9). Then for each set $B \in \mathcal{F}$, we have:

$$\int_B f dp = \int_B g dp.$$

Put $E_1 = (f < g)$ and $E_2 = (g < f)$. Obviously, E_1 and $E_2 \in \mathcal{F}$. If the inequality $p(E_1) > 0$ should hold, it would follow that

$$\int_{E_1} g dp = \int_{E_1} f dp < \int_{E_1} g dp,$$

which is impossible. This contradiction yields $p(E_1) = 0$. We can similarly show that $p(E_2) = 0$. These last two equalities imply that $p(f \neq g) = 0$, i.e. the decomposition (9) is unique. The theorem is thus proved. ■

Let $E \in \mathcal{F}$ be arbitrarily fixed with $p(E) > 0$. Consider the set function $p^* : \mathcal{F} \to [0, 1]$, defined by

$$p^*(B) = \frac{p(B \cap E)}{p(E)}$$

. Clearly, p^* is an optimal measure and $p^* \ll p$. It is evident that

$$\frac{dp^*}{dp} = \frac{\chi(E)}{p(E)} \cdot p$$

almost everywhere (by the *optimal Radon-Nikodym theorem*).

Definition 4.3.3 *The above set function $p^*(B)$ will be called conditional optimal measure of B given E, and will be denoted by $p(B|E)$.*

Definition 4.3.4 *Let f be any measurable function with $A|f| < \infty$ and $E \in \mathcal{F}$, with $p(E) > 0$. The conditional optimal average of f given E is defined by*

$$A_p(|f|\,|E) := \int_E |f|\,dp^*.$$

Lemma 4.3.2 *Let f be any measurable function with $A|f| < \infty$ and $E \in \mathcal{F}$, with $p(E) > 0$. Then*

$$A_p(|f|\,|E) := \frac{1}{p(E)} \int_E |f|\,dp.$$

4.4 The Fubini's type theorem

Let $(\Omega_i, \mathcal{F}_i, p_i)$, $i = 1, 2$, be two optimal measure spaces and let us denote the smallest σ-algebra containing $\mathcal{F}_1 \times \mathcal{F}_2$ by $\mathcal{S} := \sigma(\mathcal{F}_1 \times \mathcal{F}_2)$. For each $\omega_i \in \Omega_i$ ($i = 1, 2$), we define ω_1 *cross-section* and ω_2 *cross-section* by $E_{\omega_1} = \{\omega \in \Omega_2 : (\omega_1, \omega) \in E\}$ and $E^{\omega_2} = \{\omega \in \Omega_1 : (\omega, \omega_2) \in E\}$, where $E \in \mathcal{S}$.

Definition 4.4.1 *Let f be any measurable function defined on $(\Omega_1 \times \Omega_2, \mathcal{S})$. For each $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$, the functions*

1. $f_{\omega_1} : \Omega_2 \to \mathbb{R} \cup \{-\infty, \infty\}$ *defined by* $f_{\omega_1}(\omega_2) = f(\omega_1, \omega_2)$,

respectively

2. $f_{\omega_2} : \Omega_1 \to \mathbb{R} \cup \{-\infty, \infty\}$ *defined by* $f_{\omega_2}(\omega_1) = f(\omega_1, \omega_2)$,
will be called ω_1-section, respectively ω_2-section of function f.

Theorem 4.4.1 *For every $E \in \mathcal{S}$, define the functions*
$m_E : \Omega_1 \to [0, \infty]$ *by* $m_E(\omega_1) = p_2(E_{\omega_1})$
and
$m^E : \Omega_2 \to [0, \infty]$ *by* $m^E(\omega_2) = p_1(E^{\omega_2})$.
Then

1. m_E *is $\mathcal{F}_1$-measurable.*

2. m^E *is $\mathcal{F}_2$-measurable.*

3. $\int_{\Omega_1} m_E dp_1 = \int_{\Omega_2} m^E dp_2$.
Furthermore, define the function $p_1 \times p_2 : \mathcal{S} \to [0, 1]$ by

$$p_1 \times p_2(E) = \int_{\Omega_1} m_E dp_1 = \int_{\Omega_2} m^E dp_2.$$

Then $p_1 \times p_2$ is an optimal measure such that

$$p_1 \times p_2(B \times D) = p_1(B) \cdot p_2(D),$$

for all $B \in \mathcal{F}_1$ and $D \in \mathcal{F}_2$.

Proof. Let $\overline{\mathcal{S}}$ denote the collection of all $E \in \mathcal{S}$ for which properties 1.–3. of the theorem hold. It is enough to prove that $\overline{\mathcal{S}}$ is a σ-algebra containing $\mathcal{S}$. The proof is as in the classical case (cf. [37]) except the following claim:

For all E_1 and $E_2 \in \overline{\mathcal{S}}$, $E = E_1 \cup E_2 \in \overline{\mathcal{S}}$.

In fact, by definition and *Axiom 2.1* we can easily observe that

$$m_E(\omega_1) = \max\{m_{E_1}(\omega_1),\, m_{E_2}(\omega_1)\}$$

and

$$m^E(\omega_2) = \max\left\{m^{E_1}(\omega_2),\, m^{E_2}(\omega_2)\right\}.$$

Thus

$$\int_{\Omega_1} m_E dp_1 = \max\left\{\int_{\Omega_1} m_{E_1} dp_1\,,\ \int_{\Omega_1} m_{E_2} dp_1\right\} = \max\left\{\int_{\Omega_2} m^{E_1} dp_2\,,\ \int_{\Omega_2} m^{E_2} dp_2\right\}$$
$$= \int_{\Omega_2} m^E dp_2.$$

Hence $E = E_1 \cup E_2 \in \overline{\mathcal{S}}$, since obviously m_E (resp. m^E) is -$\mathcal{F}_1$ (resp. -$\mathcal{F}_2$) measurable, ending the proof. ∎

Theorem 4.4.2 (Optimal Fubini, [5]) *Let $(\Omega_1, \mathcal{F}_1, p_1)$ and $(\Omega_2, \mathcal{F}_2, p_2)$ be two optimal measure spaces and let $f : \Omega_1 \times \Omega_2 \to \mathbb{R} \cup \{-\infty, \infty\}$ be any measurable function such that $\int_{\Omega_1\times\Omega_2} |f|\, dp < \infty$. Then,*

1. *The ω_1-section $|f_{\omega_1}| : \Omega_2 \to [0, \infty]$ is such that $\int_{\Omega_2} |f_{\omega_1}|\, dp_2 < \infty$ almost everywhere on Ω_1. The function $\varphi : \Omega_1 \to [0, \infty]$, defined by $\varphi(\omega_1) = \int_{\Omega_2} |f_{\omega_1}|\, dp_2$, is such that $\int_{\Omega_1} \varphi dp_1 < \infty$.*

2. *The ω_2-section $|f_{\omega_2}| : \Omega_1 \to [0, \infty]$ is such that $\int_{\Omega_1} |f_{\omega_2}|\, dp_1 < \infty$ almost everywhere on Ω_2. The function $\psi : \Omega_2 \to [0, \infty]$, defined by $\psi(\omega_2) = \int_{\Omega_1} |f_{\omega_2}|\, dp_1$, is such that $\int_{\Omega_2} \psi dp_2 < \infty$.*

3. *Furthermore,*

$$\int_{\Omega_1\times\Omega_2} |f|\, d(p_1 \times p_2) = \int_{\Omega_1}\left(\int_{\Omega_2} |f|\, dp_2\right) dp_1 = \int_{\Omega_2}\left(\int_{\Omega_1} |f|\, dp_1\right) dp_2$$

We shall not prove the *optimal Fubini* theorem. Instead, we simply note that the proof follows from *Theorem 4.4.1* using the same techniques as in the proof of the original *Fubini theorem*, cf. [37].

CHAPTER V

CONVERGENCE THEOREMS RELATED TO MEASURABLE FUNCTIONS

5.1 Introduction

Definition 5.1.1 *Let X be an arbitrary nonempty set. We say that a sequence of real-valued functions (h_n) converges to a real-valued function h:*

i. *discretely if for every $x \in X$ there exists a positive integer $n_0(x)$ such that $h_n(x) = h(x)$, whenever $n > n_0(x)$;*

ii. *equally if there is a sequence (b_n) of positive numbers tending to 0 and for every $x \in X$ there can be found an index $n_0(x)$ such that $|h_n(x) - h(x)| < b_n$ whenever $n > n_0(x)$.*

For more about these notions, see [27, 28, 29].

In this section we shall characterize the discrete, equally, pointwise and uniformly convergence theorems. We can say that the notions of the pointwise and uniformly convergence is ancient.

5.2 Convergence with respect to individual optimal measures

In the present section we shall be dealing with an arbitrarily fixed optimal measure space $(\Omega, \mathcal{F}, p)$, unless otherwise stated.

The following three results are the counterparts of the monotone convergence theorem, the Fatou lemma and the dominated convergence theorem in measure theory.

Theorem 5.2.1 (Optimal monotone convergence, [5])

1. *If (f_n) is an increasing sequence of non-negative measurable functions, then*

$$\lim_{n\to\infty} \bigvee_{\Omega} f_n dp = \bigvee_{\Omega} \left(\lim_{n\to\infty} f_n\right) dp.$$

2. *If (g_n) is a decreasing sequence of non-negative measurable functions with $g_1 \leq b$ for some $b \in (0,\ \infty)$, then*

$$\lim_{n\to\infty} \bigvee_{\Omega} g_n dp = \bigvee_{\Omega} \left(\lim_{n\to\infty} g_n\right) dp.$$

We shall here below give an example showing the reason why the optimal monotone convergence theorem fails to hold for all decreasing sequences of measurable functions.

Example 5.2.1 *Let* $(\mathbb{N}, \mathcal{F}, p)$ *be the optimal measure space we considered in Example 4.2.1. Define the following measurable function*

$$g_n(\omega) = \begin{cases} 0 & \text{if } \omega < n \\ \omega & \text{if } \omega \geq n. \end{cases}$$

Obviously, (g_n) *tends decreasingly to zero as* $n \to \infty$*. Let us show that* $Ag_n = 1$ *for all* $n \in \mathbb{N}$*. Obviously,* $Ag_n \geq np(\{n\}) = 1$*. On the other hand, let* $0 \leq s \leq g_n$ *where* $s = \sum_{j=1}^{k} b_j \chi(B_j)$*. Denote* $\omega_j = \min B_j$ *for* $j = 1, \ldots, k$*. Then* $p(B_j) = \frac{1}{\omega_j}$ *and* $b_j \leq \omega_j$ *for all* $j = 1, \ldots, k$*. Hence inequality* $b_j p(B_j) \leq 1$ *holds for each index* $j = 1, \ldots, k$*. Consequently,* $I(s) \leq 1$, $0 \leq s \leq g_n$*. Thus* $Ag_n \leq 1$ *whenever* $n \in \mathbb{N}$*.*

Lemma 5.2.1 (Optimal Fatou, [5]) *If* (f_n) *and* (h_n) *are sequences of non-negative measurable functions, then for every optimal measure p, we have that:*

1. $\bigvee_{\Omega} \left(\liminf_{n\to\infty} f_n\right) dp \leq \liminf_{n\to\infty} \bigvee_{\Omega} f_n dp$;

2. $\limsup_{n\to\infty} \bigvee_{\Omega} h_n dp \leq \bigvee_{\Omega} \left(\limsup_{n\to\infty} h_n\right) dp$, *whenever* (h_n) *is a uniformly bounded sequence.*

Proof. To prove the first part, we point out by definition that

$$\liminf_{n\to\infty} f_n = \max\{\min\{f_k : k \geq n\} : n \in \mathbb{N}\}.$$

Let $f_n^* = \min\{f_k : k \geq n\}$, $n \in \mathbb{N}$, and $f = \liminf_{n\to\infty} f_n$. Clearly, (f_n^*) is an increasing sequence. The *optimal monotone convergence theorem* implies that

$$A\left(\liminf_{n\to\infty} f_n\right) \leq \liminf_{n\to\infty} Af_n.$$

To end the proof we note that the second part can be similarly verified. ■

Theorem 5.2.2 *Let* (f_n) *be a uniformly bounded sequence of non-negative measurable functions. Then* $A(\lim_{n\to\infty} f_n) = Af$*, where* $\lim_{n\to\infty} f_n = f$ *almost everywhere.*

Proof. The *optimal Fatou lemma* via the assumption implies that

$$A\left(\liminf_{n\to\infty} f_n\right) \leq \liminf_{n\to\infty} Af_n \leq \limsup_{n\to\infty} Af_n \leq A\left(\limsup_{n\to\infty} f_n\right).$$

By assumption $f = \liminf_{n\to\infty} f_n = \limsup_{n\to\infty} f_n$ almost everywhere. Consequently,

$$Af \leq \liminf_{n\to\infty} Af_n \leq A\left(\limsup_{n\to\infty} f_n\right) \leq Af$$

meaning that $\lim_{n\to\infty} Af_n = Af$. This was to be proven. ■

Lemma 5.2.2 *Let $\omega \in \Omega$ be fixed. Then for every measurable function f, we have that $z_f(p_\omega) = |f(\omega)|$.*

Proof. Let $0 \leq s = \sum_{i=1}^{k} b_i \chi(B_i)$ be a measurable simple function. Then it is obvious that $z_s(p_\omega) = s(\omega)$. Let (s_n) be a sequence of non-negative measurable simple functions tending increasingly to $|f|$. Then by *Theorem 5.2.1* it ensues that

$$z_f(p_\omega) = \lim_{n\to\infty} z_{s_n}(p_\omega) = \lim_{n\to\infty} s_n(\omega) = |f(\omega)|$$

which was to be proved. ■

5.3 Characterization of various types of convergence for measurable functions

We say that a nonempty measurable set E is closely related to some sequence $(\omega_n) \subset \Omega$ if

$$|E \cap \{\omega_n : n \in \mathbb{N}\}| = \begin{cases} \infty, & \text{if } |E| = \infty \\ |E|, & \text{if } |E| < \infty, \end{cases}$$

that is, if E is infinite, then infinitely many members of the sequence belong to E, otherwise all of its elements are members of the sequence.

Definition 5.3.1 *Let E be closely related to a sequence $(\omega_n) \subset \Omega$, and let $(\alpha_n) \subset [0, 1]$ be any fixed sequence tending decreasingly to 0. The optimal measure $p_E : \mathcal{F} \to [0, 1]$, defined by $p_E(B) = \max\{\alpha_n : \omega_n \in B\}$, will be called 1st-type E-dependent optimal measure.*

Theorem 5.3.1 (Agbeko, [7]) *Let f and f_n $(n \in \mathbb{N})$ be any measurable functions. Then (f_n) tends to f uniformly if and only if (z_n) tends to 0 uniformly on $\mathcal{P}_\infty$, where $z_n(p) = \bigvee_\Omega |f_n - f|\, dp$ with $n \in \mathbb{N}$, $p \in \mathcal{P}_\infty$.*

Proof. *Sufficiency.* Suppose that (z_n) tends to 0 uniformly. To prove the sufficiency it is enough to show that for every number $b > 0$, there can be found some $n_0(b) \in \mathbb{N}$ such that $(|f - f_n| \geq b) = \emptyset$ whenever $n \geq n_0(b) + 1$. In fact, let us assume that the contrary holds. Then for some $b_0 > 0$ and all $n_0 \in \mathbb{N}$, there is an integer $m > n_0$ such that $(|f - f_m| \geq b_0) \neq \emptyset$. Define

$$n_1 = \min\{m > n_1 : (|f - f_m| \geq b_0) \neq \emptyset\}$$

when $n_0 = 1$. If n_k has been selected, define

$$n_{k+1} = \min\{m > n_k : (|f - f_m| \geq b_0) \neq \emptyset\}$$

when $n_0 = n_k$. It is clear that sequence (n_k) tends increasingly to infinity alongside with k, so that $(|f - f_{n_k}| \geq b_0) \neq \emptyset$, $k \in \mathbb{N}$. Then by assumption some $n_m \in \{n_k : k \in \mathbb{N}\}$

exists such that $z_{n_k}(p) < b_0$, for all $k \geq m$ and $p \in \mathcal{P}_\infty$. Now let $E_m = \bigcup_{k=m}^{\infty} B_{n_k}$, (where $B_{n_k} = (|f - f_{n_k}| \geq b_0)$, $k \in \mathbb{N}$). Write $H_{n_m} = B_{n_m}$ and for $k \geq m+1$, set $H_{n_k} = \left(\bigcup_{j=m}^{k} B_{n_j}\right) \setminus \left(\bigcup_{j=m}^{k-1} B_{n_j}\right)$. Clearly, $\mathcal{H} = \{H_{n_k} : k \geq m\}$ is a sequence of pairwise disjoint measurable sets with $E_m = \bigcup_{k=m}^{\infty} H_{n_k}$. Fix a sequence $(\omega_k) \in \Omega$ so that $\omega_k \in H_{n_k}$ whenever $k \geq m$. Next, let $p_0 \in \mathcal{P}_\infty$ be a 1st-type E_m-dependent optimal measure defined by $p_0(B) = n_m \cdot \max\left\{\frac{1}{n_k} : \omega_k \in B\right\}$. It is obvious that $\mathcal{H}$ is a p_0-generating system. Hence we have on one hand that $z_{n_m}(p_0) < b_0$. Nevertheless on the other hand we also obtain that $z_{n_m}(p_0) \geq \int_{H_{n_m}} |f_{n_m} - f| \, dp_0 \geq b_0$, since $p_0(H_{n_m}) = 1$. As these last two inequalities contradict each other, the sufficiency is thus proved.
Necessity. Assume that $f_n \to f$ uniformly, as $n \to \infty$. Then for every $b \in (0, \infty)$, there is some $n_0(b) \in \mathbb{N}$ such that $\left(|f_n - f| < \frac{b}{2}\right) = \Omega$ whenever $n > n_0(b)$. Consequently, for every $p \in \mathcal{P}_\infty$, it ensues that $z_n(p) \leq \frac{b}{2} < b$, $n > n_0(b)$. This completes the proof of the theorem. ∎

Lemma 5.3.1 *Let f and f_n $(n \in \mathbb{N})$ be any measurable functions. If (f_n) tends to f pointwise (equally or discretely), then $\limsup_{n\to\infty} B_n = \emptyset$, where $B_n = (|f_n - f| = \infty)$, $n \in \mathbb{N}$.*

Proof. It is enough to prove the lemma for the pointwise convergence, since the proof of the remaining cases is similarly done. Assume that $E := \limsup_{n\to\infty} B_n \neq \emptyset$. Let us pick an arbitrary $\omega \in E$. Then it is clair that $\limsup_{n\to\infty} |f_n(\omega) - f(\omega)| = \infty$ and hence $\bigwedge_{n=k}^{\infty} \bigvee_{j=n}^{\infty} |f_j(\omega) - f(\omega)| = \infty$ for every $k \in \mathbb{N}$. But since (f_n) tends to f pointwise we must have that for every constant $b > 0$ there is a positive integer $m_0 = m_0(b, \omega)$ such that $|f_n(\omega) - f(\omega)| < b$ whenever $n > m_0$. Hence $b \geq \bigwedge_{n=m_0}^{\infty} \bigvee_{j=n}^{\infty} |f_j(\omega) - f(\omega)| = \infty$, which is absurd, completing the proof. ∎

Theorem 5.3.2 (Agbeko, [7]) *Let (f_n) be any sequence of measurable functions. Then (f_n) tends to a measurable function f pointwise if and only if (z_n) tends to 0 pointwise on $\mathcal{P}_{<\infty}$, where for every $n \in \mathbb{N}$, z_n is defined on $\mathcal{P}_{<\infty}$ by*

$$z_n(p) = \int_\Omega |f_n - f| \, dp.$$

Proof. *Sufficiency.* Assume that for all $b > 0$ and $p \in \mathcal{P}_{<\infty}$ there is a positive integer $n_0 = n_0(b, p)$ such that $z_n(p) < b$ whenever $n > n_0$. Then since for every fixed $\omega \in \Omega$ the ω-concentrated measure p_ω depends solely upon $\omega \in \Omega$, index $n_0(b, p_\omega)$ also depends on ω. Hence via *Lemma 5.2.2* we have for all $n \geq n_0(b, \omega) = n_0(b, p_\omega)$ that $|f_n(\omega) - f(\omega)| = z_n(p_\omega) < b$.
Necessity. Suppose that for all $a > 0$ and $\omega \in \Omega$, there can be found some positive integer

$m_0 = m_0(a, \omega)$ such that $|f_n(\omega) - f(\omega)| < a$, whenever $n \geq m_0$. Assume further that there is some $b > 0$ and some $p \in \mathcal{P}_{<\infty}$ such that for every $n \in \mathbb{N}$, there exists some $m \geq n$ with the property that $z_n(p) \geq b$. Let $H_1, \ldots, H_k$ be a p-generating system. Via *Lemma 5.3.1*, there is some $n_0 \in \mathbb{N}$, big enough so that $f_n - f$ is finite on Ω whenever $n \geq n_0$. Then for every $n \geq n_0$, a measurable set $A_n^{(i)}$ exists with $A_n^{(i)} \subset H_i$ and $p\left(A_n^{(i)}\right) = 0$ such that $f_n - f$ is constant on $H_i \backslash A_n^{(i)}$, $i = 1, \ldots, k$ (because of *Remark* 4.2.1). Clearly, $p\left(\bigcup_{j=n_0}^{\infty} A_j^{(i)}\right) = 0$, so that the identity $p\left(H_i \backslash \bigcup_{j=n_0}^{\infty} A_j^{(i)}\right) = p(H_i)$ holds. Hence $f_n - f$ is constant on $H_i \backslash \bigcup_{j=n_0}^{\infty} A_j^{(i)}$ whenever $i \in \{1, \ldots, k\}$ and $n \geq n_0$. Fix $\omega_i \in H_i \backslash \bigcup_{j=n_0}^{\infty} A_j^{(i)}$, $i \in \{1, \ldots, k\}$. Then by assumption there must be some positive integer $k_0^{(i)} = k_0(b, \omega_i)$ such that $|f_n(\omega_i) - f(\omega_i)| < b$, $n > k_0^{(i)}$. Thus for all $n \geq k_0$ (where $k_0 = \bigvee_{i=1}^{k} k_0^{(i)}$), we have that $\bigvee_{i=1}^{k} |f_n(\omega_i) - f(\omega_i)| < b$. Now write $k^* = \max(k_0, n_0)$. Then some integer $m > k^*$ exists such that $z_m(p) \geq b$. Therefore, via *Proposition 4.2.3* and *Remark 4.2.1*, we obtain that

$$b \leq z_m(p) = \bigvee_{i=1}^{k} c_i \cdot p(H_i) \leq \bigvee_{i=1}^{k} c_i = \bigvee_{i=1}^{k} |f_m(\omega_i) - f(\omega_i)| < b$$

where for $i \in \{1, \ldots, k\}$, $c_i = |f_m(\omega) - f(\omega)|$ if $\omega \in H_i \backslash \bigcup_{j=n_0}^{\infty} A_j^{(i)}$. However, this is absurd, a contradiction which ends the proof of the theorem. ■

Theorem 5.3.3 (Agbeko, [7]) *A sequence of measurable functions (f_n) converges to some measurable function f equally if and only if (z_n) converges to 0 equally on $\mathcal{P}_{<\infty}$, where for every $n \in \mathbb{N}$, z_n is defined on $\mathcal{P}_{<\infty}$ by $z_n(p) = \int_\Omega |f_n - f|\, dp$.*

Proof. *Necessity.* Suppose that there exists a sequence $(b_n) \subset (0, \infty)$ tending to 0 and for every $\omega \in \Omega$ there can be found a positive integer $n_0(\omega)$ such that $|f_n(\omega) - f(\omega)| < b_n$ for all $n \geq n_0(\omega)$. It is enough to show that the equal convergence of (z_n) holds true for this sequence (b_n). In fact, assume that for this sequence (b_n), there is some $p \in \mathcal{P}_{<\infty}$ such that for all $j \in \mathbb{N}$ an integer $m = m(p) > j$ can be found with the property that $z_m(p) \geq b_m$. Let $H_1, \ldots, H_k$ be a p-generating system. Via *Lemma 5.3.1*, there is some $n_0 \in \mathbb{N}$, big enough so that $f_n - f$ is finite on Ω whenever $n \geq n_0$. Then for every $n \geq n_0$, a measurable set $A_n^{(i)}$ exists with $A_n^{(i)} \subset H_i$ and $p\left(A_n^{(i)}\right) = 0$ such that $f_n - f$ is constant on $H_i \backslash A_n^{(i)}$, $i = 1, \ldots, k$. But as $p\left(\bigcup_{j=n_0}^{\infty} A_j^{(i)}\right) = 0$, we can easily observe that $p\left(H_i \backslash \bigcup_{j=n_0}^{\infty} A_j^{(i)}\right) = p(H_i)$, $i \in \{1, \ldots, k\}$. Hence $f_n - f$ is constant on $H_i \backslash \bigcup_{j=n_0}^{\infty} A_j^{(i)}$ for all $i \in \{1, \ldots, k\}$ and $n \geq n_0$. Fix $\omega_i \in H_i \backslash \bigcup_{j=n_0}^{\infty} A_j^{(i)}$, $i \in \{1, \ldots, k\}$. Then by assumption there must be some positive integer $k_0^{(i)} = k_0(\omega_i)$ such that $|f_n(\omega_i) - f(\omega_i)| < b_n$, $n > k_0^{(i)}$. Thus for all $n \geq k_0$ (where $k_0 = \bigvee_{i=1}^{k} k_0^{(i)}$), we have that $\bigvee_{i=1}^{k} |f_n(\omega_i) - f(\omega_i)| < b_n$. Consequently, we have on one hand that $z_m(p) \geq b_m$. But on the other hand, *Proposition 4.2.3* yields that

$$z_m(p) = \bigvee_{i=1}^{k} c_i \cdot p(H_i) \leq \bigvee_{i=1}^{k} c_i = \bigvee_{i=1}^{k} |f_m(\omega_i) - f(\omega_i)| < b_m$$

(where for $i \in \{1, \ldots, k\}$, $c_i = |f_m(\omega) - f(\omega)|$ if $\omega \in H_i \setminus \bigcup_{j=n_0}^{\infty} A_j^{(i)}$), meaning that $b_m < b_m$, which is however absurd. This contradiction concludes the proof of the necessity. *Sufficiency.* Assume that there is a sequence (b_n) of positive numbers tending to 0 and for every $p \in \mathcal{P}_{<\infty}$ there exists a positive integer $n_0(p)$ such that $z_n(p) < b_n$ whenever $n > n_0(p)$. Then for each fixed $\omega \in \Omega$, *Lemma 5.2.2* entails that $|f_n(\omega) - f(\omega)| = z_n(p_\omega) < b_n$ whenever $n > n_0(p_\omega) = n_0(\omega)$. The sufficiency is thus proved, which completes the proof of the theorem. ■

Theorem 5.3.4 (Agbeko, [7]) *A sequence of measurable functions (f_n) converges to some measurable function f discretely if and only if (z_n) converges to 0 discretely on $\mathcal{P}_{<\infty}$, where for every $n \in \mathbb{N}$, z_n is defined on $\mathcal{P}_{<\infty}$ by $z_n(p) = \int_\Omega |f_n - f| \, dp$.*

The proof for *Theorem 5.3.4* is omitted because it can be carried out "mutatis mutandis" as done in *Theorems 5.3.2* and *5.3.3*.

Definition 5.3.2 *A sequence of measurable functions (f_n) is said to converge in optimal measure to some measurable function f if $\lim_{n\to\infty} p(|f_n - f| \geq \varepsilon) = 0$ for every constant $\varepsilon > 0$.*

Theorem 5.3.5 (Agbeko, [5]) *Let (f_n) be a sequence of measurable functions (f_n) which converges in optimal measure to some measurable function f. Then there exists a subsequence (f_{n_k}) which converges to f almost everywhere.*

5.4 Characterization of various types of boundedness

Remark 5.4.1 *If (x_n) is a sequence of real numbers such that $\limsup_{n\to\infty} |x_n| < \infty$, then for each of its subsequences (x_{n_k}) we have that $\limsup_{k\to\infty} |x_{n_k}| < \infty$.*

Notice. For every fixed measurable function f, the mapping $z_f : \mathcal{P} \to [0, \infty]$, defined by $z_f(p) = \int_\Omega |f| \, dp$, is a function.

Lemma 5.4.1 *Let $\omega \in \Omega$ be fixed. Then for every measurable function f, we have that $z_f(p_\omega) = |f(\omega)|$.*

Proof. Let $0 \le s = \sum_{i=1}^{k} b_i \chi(B_i)$ be a measurable simple function. Then it is obvious that $z_s(p_\omega) = s(\omega)$. Let (s_n) be a sequence of non-negative measurable simple functions tending increasingly to $|f|$. Then by *Theorem B* it ensues that

$$z_f(p_\omega) = \lim_{n\to\infty} z_{s_n}(p_\omega) = \lim_{n\to\infty} s_n(\omega) = |f(\omega)|$$

which was to be proved. ■

Theorem 5.4.1 (Agbeko, [7]) *Let f be any measurable function the following assertions are equivalent.*

1. *f is bounded.*

2. $\lim_{x\to\infty} \bigvee_{(|f|\ge x)} |f|\, dp = 0$ *for all* $p \in \mathcal{P}_\infty$.

3. *There exists a constant $b > 0$ such that* $\bigvee_{\Omega} |f|\, dp \ne b$ *for all* $p \in \mathcal{P}_\infty$.

The proof will be carried out in two steps. In *Proposition 5.4.1* we shall show the equivalence 1. $\iff$ 2. and then the equivalence $\rceil 1.\iff \rceil 3.$ in *Proposition 5.4.2.*

Proposition 5.4.1 *A measurable function f is bounded if and only if*

$$\lim_{x\to\infty} \bigvee_{(|f|\ge x)} |f|\, dp = 0$$

for all $p \in \mathcal{P}_\infty$.

Proof. Suppose that f is bounded, and write $b > 0$ for its bound. Then for every $p \in \mathcal{P}_\infty$, we have that $\bigvee_{(|f|\ge x)} |f|\, dp \le b \cdot p(|f| \ge x) \to 0$, as $x \to \infty$.
Conversely, assume that $\lim_{k\to\infty} \bigvee_{(|f|\ge k)} |f|\, dp = 0$ *for all* $p \in \mathcal{P}_\infty$, but for every $n \in \mathbb{N}$ we have that $(|f| \ge n-1) \ne \emptyset$. It obviously ensues that

$$(|f| \ge n-1) \setminus (|f| \ge n) = H_n \ne \emptyset$$

for infinitely many $n \in \mathbb{N}$. Suppose without loss of generality that $H_n \ne \emptyset$, $n \in \mathbb{N}$. Further let $(\omega_n) \subset \Omega$ be such that $\omega_n \in H_n$ for all $n \in \mathbb{N}$. Define $p \in \mathcal{P}_\infty$ by $p(B) = \max\left\{\frac{1}{n} : \omega_n \in B\right\}$. Clearly, (H_n) is a generating system for p. Then by assumption it follows that $\lim_{k\to\infty} \bigvee_{(|f|\ge k)} |f|\, dp = 0$. Now note that $(|f| \ge k) = \bigcup_{i=k+1}^{\infty} H_i$ for all $k \in \mathbb{N}$. Hence Proposition 4.3.2 entails that $\bigvee_{(|f|\ge k)} |f|\, dp = \sup_{i\ge k+1} \bigvee_{H_i} |f|\, dp$. It is not difficult to check that $\bigvee_{H_i} |f|\, dp \ge 1 - \dfrac{1}{i}$, $i \ge k+1$. Consequently, it results that $\bigvee_{(|f|\ge k)} |f|\, dp \ge 1 - \dfrac{1}{k+1}$ $(k \in \mathbb{N})$, leading to $0 = \lim_{k\to\infty} \bigvee_{(|f|\ge k)} |f|\, dp \ge 1$, which is absurd. This contradiction concludes on the validity of the sufficiency, ending the proof. ■

Proposition 5.4.2 *Let f be a finite measurable function. Then f is unbounded if and only if for every constant $c > 0$, there exists some $p_c \in \mathcal{P}_\infty$ such that*

$$z_f(p_c) = c. \tag{10}$$

Proof. *Necessity.* Assume that f is unbounded measurable function. For every $n \in \mathbb{N}$, write $E_n = (c \cdot (n-1) \le |f| < c \cdot n)$, where $c > 0$ is an arbitrarily fixed constant. Clearly, the members of the sequence (E_n) are pairwise disjoint and $\Omega = \bigcup_{n=1}^{\infty} E_n$. Fix a sequence $(\omega_n) \subset \Omega$ in the following way: $\omega_n \in E_n$, $n \in \mathbb{N}$. Define $p_c \in \mathcal{P}_\infty$ by $p_c(B) = \max\left\{\frac{1}{n} : \omega_n \in B\right\}$. It is obvious that sequence (E_n) is a p_c-generating system such that $z_f(p_c) = \sup_{n \ge 1} \int_{E_n} |f|\, dp_c$, because of Proposition 4.3.2. But as $\left(1 - \frac{1}{n}\right) c \le \int_{E_n} |f|\, dp_c < c$ (for all $n \in \mathbb{N}$), it ensues that $c = \sup_{n \ge 1} \int_{E_n} |f|\, dp_c = z_f(p_c)$.
Sufficiency. Suppose that for every constant $c > 0$, identity (*10*) holds with a suitable $p \in \mathcal{P}_\infty$. Assume that f is bounded (and denote by b its bound). Now let $c > b$ be any fixed constant with a corresponding $p_c \in \mathcal{P}_\infty$ satisfy (*10*). Then we trivially obtain that $z_f(p_c) \le b$. Hence we must have that $c \le b$, which is in contradiction with the choice of c. This absurdity allows us to conclude on the validity of the proposition. ■

Lemma 5.4.2 *Let $p \in \mathcal{P}_\infty$ and (B_n) be a sequence of measurable sets tending increasingly to a measurable set $B \neq \emptyset$. Then there exists some $n_0 \in \mathbb{N}$ such that $p(B) = p(B_n)$ whenever $n \ge n_0$.*

The proof given to *Lemma 0.1*, [5], is also valid for *Lemma 5.4.2*, so we shall omit it.

We shall next give a set of measurable functions, including the uniformly bounded ones, which we shall proceed to characterize latter on.

Definition 5.4.1 *We say that a sequence of measurable functions (f_n) is uniformly bounded starting from an index if there can be found a real number $b > 0$ and some positive integer n_0 such that $(f_n > b) = \emptyset$ for all integers $n > n_0$. (We shall simply say that (f_n) is i-uniformly bounded.)*

The following two results are just the extensions of *Theorem 5.2.1* and *Lemma 5.2.1.* We shall omit their proofs as they can be similarly carried out.

Lemma 5.4.3 *Let (g_n) be a decreasing sequence of non-negative measurable functions and $\lim_{n\to\infty} g_n = g$ such that $(g_m \le b) = \Omega$ for some $m \ge 1$ and some constant $b > 0$. Then for all $p \in \mathcal{P}$*

$$\lim_{n\to\infty} \int_\Omega g_n dp = \int_\Omega g dp.$$

Lemma 5.4.4 *Let (f_n) be an i-uniformly bounded sequence of non-negative measurable functions. Then for every $p \in \mathcal{P}$*

$$\limsup_{n\to\infty} \int_\Omega f_n dp \leq \int_\Omega \left(\limsup_{n\to\infty} f_n\right) dp.$$

Theorem 5.4.2 (Agbeko, [7]) *Let (f_n) be an arbitrary sequence of measurable functions. Then*

1. *Sequence (f_n) is i-uniformly bounded, if and only if the following two assertions hold simultaneously:*
2. *$z_f(p) \leq c$ for some constant $c > 0$ and all $p \in \mathcal{P}_\infty$;*
3. *$\limsup_{n\to\infty} z_n(p) \leq z_f(p)$, for all $p \in \mathcal{P}_\infty$ (where $f = \limsup_{n\to\infty} |f_n|$ and $z_n(p) = \int_\Omega |f_n| dp$ with $n \in N$, $p \in \mathcal{P}_\infty$).*

Proof. *Necessity.* We just note that the implication *1.→ 2.* is obvious and on the other hand the implication *1.→ 3.* is no more than *Lemma 5.4.4*.
Sufficiency. Assume that assertions *2.* and *3.* hold simultaneously. Let us suppose further that assertion *1.* is false, i.e. for every real number $b > 0$ and any positive integer n_0 there is some integer $m > n_0$ such that $(|f_m| > b) \neq \emptyset$. Then we can choose by recurrence a sequence (n_k) of positive integers as follows. Write $n_1 = 1$ and $n_2 = \min\{m > n_1 : (|f_m| > n_1) \neq \emptyset\}$. If n_k has been defined, then write $n_{k+1} = \min\{m > n_k : (|f_m| > k \cdot n_k) \neq \emptyset\}$. Clearly, the sequence (n_k) tends increasingly to infinity and for all positive integers $k \in \mathbb{N}$, $\left(\left|f_{n_{k+1}}\right| > k \cdot n_k\right) \neq \emptyset$. Now set $E = \bigcup_{k=1}^{\infty} B_{n_k}$, where $B_{n_k} = \left(\left|f_{n_{k+1}}\right| > k \cdot n_k\right)$, $k \in \mathbb{N}$. Write $H_1 = B_{n_1}$, and $H_k = \left(\bigcup_{j=1}^{k} B_{n_j}\right) \setminus \left(\bigcup_{j=1}^{k-1} B_{n_j}\right)$, $k > 2$. It is obvious that (H_k) is a sequence of pairwise disjoint measurable sets with $E = \bigcup_{k=1}^{\infty} H_k$. Let $p \in \mathcal{P}_\infty$ be a *1st-type E-dependent* optimal measure defined by $p(B) = \max\left\{\frac{1}{k} : \omega_k \in B\right\}$, where $(\omega_k) \subset \Omega$ is a fixed sequence so that $\omega_k \in H_k$ $(k \in \mathbb{N})$. It is clear that $\mathcal{H}(p) = \{H_k : k \in \mathbb{N}\}$ is a p-generating system. Then via assertions 2. and 3. we have that $c \geq \int_\Omega \left(\limsup_{n\to\infty} |f_n|\right) dp \geq \limsup_{n\to\infty} \int_\Omega |f_n| dp$ and hence $b > \limsup_{k\to\infty} \int_\Omega \left|f_{n_{k+1}}\right| dp$ for some $b > 0$ (this is true because of *Remark 5.4.1*). Consequently, as $p(H_k) = \dfrac{1}{k}$ for every $k \in \mathbb{N}$, we must have

$$\begin{aligned} b &> \limsup_{k\to\infty} \int_\Omega \left|f_{n_{k+1}}\right| dp = \limsup_{k\to\infty} \int_E \left|f_{n_{k+1}}\right| dp \geq \limsup_{k\to\infty} \int_{H_k} \left|f_{n_{k+1}}\right| dp \\ &\geq \limsup_{k\to\infty} k \cdot n_k \cdot p(H_k) = \infty, \end{aligned}$$

which is absurd. This contradiction justifies the validity of the argument. ■

5.5 Banach spaces induced by optimal measures

Throughout this section we shall be dealing with an arbitrary but fixed optimal measure space $(\Omega, \mathcal{F}, p)$.

Let $f : \Omega \to \mathbb{R} \cup \{-\infty, \infty\}$ be any measurable function. We shall say that f belongs to:

1. $\mathcal{A}^\infty$ if $p(|f| \leq b) = 1$ for some constant $b \in (0, \infty)$.

2. $\mathcal{A}^\alpha$ if $\bigwedge_\Omega |f|^\alpha \, dp < \infty$, $\alpha \in [1, \infty)$.

For any $\alpha \in [1, \infty]$, the space $\mathcal{A}_\alpha(p)$ endowed with the norm $\|\cdot\|_\alpha$, defined by

$$\|f\|_{\mathcal{A}^\alpha} = \begin{cases} \inf\{b \in (0, \infty) : p(|f| \leq b) = 1\}, & \text{if } f \in \mathcal{A}_\infty(p), \alpha = \infty \\ \sqrt[\alpha]{\bigwedge_\Omega |f|^\alpha \, dp}, & \text{if } f \in \mathcal{A}_\alpha(p), \alpha \in [1, \infty) \end{cases}$$

As in the case of L^p-spaces in measure theory, it can be similarly seen that $\|\cdot\|_\alpha$ is a semi-norm.

Lemma 5.5.1 **1.** *$A|fg| \leq \|f\|_{\mathcal{A}^\alpha} \|g\|_{\mathcal{A}^\infty}$ whenever $f \in \mathcal{A}^1$ and $g \in \mathcal{A}^\infty$.*

2. *Let α and $\beta \in (1, \infty)$ be such that $\alpha^{-1} + \beta^{-1}$. Then $A|fg| \leq \|f\|_{\mathcal{A}^\alpha} \|g\|_{\mathcal{A}^\beta}$ (called the optimal Hölder inequality), whenever $f \in \mathcal{A}^\alpha$ and $g \in \mathcal{A}^\beta$.*

3. *$\|f+g\|_{\mathcal{A}^\alpha} \leq \|f\|_{\mathcal{A}^\alpha} + \|g\|_{\mathcal{A}^\alpha}$ (called the optimal Minkowski inequality) whenever $f \in \mathcal{A}^\alpha$ and $g \in \mathcal{A}^\alpha$, with $\alpha \in [1, \infty]$.*

Definition 5.5.1 *Let $(f_n) \subset \mathcal{A}^\alpha$, $\alpha \in [1, \infty]$, be any sequence of measurable functions.*

1. *We say that (f_n) is a Cauchy sequence in $\mathcal{A}^\alpha$ if for every number $\varepsilon \in (0, 1)$ there is some index $n_0 := n_0(\varepsilon)$ such that $\|f_n - f_m\|_{\mathcal{A}^\alpha} < \varepsilon$, whenever $n, m \geq n_0$.*

2. *We say that (f_n) converges to a measurable function f in $\mathcal{A}^\alpha$-norm if for every number $\varepsilon \in (0, 1)$ there is some index $n_0 := n_0(\varepsilon)$ such that $\|f_n - f\|_{\mathcal{A}^\alpha} \to 0$ as $n \to \infty$.*

Remark 5.5.1 *For every number $\alpha \in (1, \infty)$, we have $\mathcal{A}^\infty \subset \mathcal{A}^\alpha \subset \mathcal{A}^1$.*

Theorem 5.5.1 (Agbeko, [5]) *For each number $\alpha \in [1, \infty]$, $\mathcal{A}^\alpha$ is a Banach space (i.e. every Cauchy sequence in $\mathcal{A}^\alpha$ converges to a measurable function in $\mathcal{A}^\alpha$-norm).*

CHAPTER VI

SOME MAXIMAL INEQUALITIES RELATED WITH PROBABILITY MEASURE

6.1 Introduction

Some notations.
* ∇:= the set of all non-negative increasing function.
* Δ_2:= the set of all convex Young functions that satisfy the growth condition.
* $\mathcal{Y}_{\text{conc}}$:= the set of all concave Young functions.
* Δ:= the set of all convex functions.

In the whole chapter we shall be dealing with an arbitrary probability space $(\Omega,\, \mathcal{F},\, P)$.

Let Φ be a *convex Young function*, i.e.

$$\Phi\left(x\right)=\int\limits_{0}^{x}\varphi\left(t\right)dt,\; x\in\mathbb{R}_{+},$$

where $\varphi:(0,\,\infty)\rightarrow(0,\,\infty)$ is a right-continuous and increasing function such that $\varphi\left(0\right)\geq 0$ and $\varphi\left(\infty\right)=\infty$. We say that Φ satisfies the *growth condition* if

$$\Phi\left(2x\right)\leq c\Phi\left(x\right)\;\text{ for all }x\in\mathbb{R}_{+}, \tag{11}$$

where c is a positive constant. For more about convex Young function, see [40].

It is also interesting to remind some facts about the so-called the conjugate Young functions, which can be defined as follows:

For $t\in(0,\,\infty)$ put $\gamma\left(t\right):=\sup\left\{x>0:\varphi\left(x\right)<t\right\}$ and let $\gamma\left(0\right)=0$. It can be easily checked that γ satisfies all the conditions imposed on φ and we trivially have $\gamma\left(\varphi\left(x\right)\right)\leq x\leq\gamma\left(\varphi\left(x\right)+0\right)$, whenever $x\in(0,\,\infty)$.

The convex Young function

$$\Gamma\left(x\right):=\int_{0}^{x}\gamma\left(t\right)dt,\;\; x\in[0,\,\infty)\,,$$

is said to be *conjugate* to Φ and the pair $(\Phi,\,\Gamma)$ is referred to as *mutually conjugate convex Young functions.*

The pair $(\Phi,\,\Gamma)$ of mutually conjugate convex Young functions satisfies the following Young inequality

$$xy\leq\Phi\left(x\right)+\,\Gamma\left(y\right)$$

for all $x,\,y\,\in\,[0,\,\infty)$, and the equality holds if and only if $y\,\in\,\left[\varphi\left(x\right),\,\varphi\left(x+0\right)\right]$ or $x\in\left[\gamma\left(y\right),\,\gamma\left(y+0\right)\right]$.

The quantity $q := \limsup_{x\to\infty} \dfrac{x\gamma(x)}{\Gamma(x)}$ is called the *power* of Γ, where Γ is a convex Young function.

Mogyoródi J. and Móri T. F. in [46] obtained the following nice result:

Theorem 6.1.1 *Let (Φ, Γ) be a pair of mutually conjugate convex Young functions. In order that the power q of Γ be finite it is necessary and sufficient that the condition*

$$\beta := \limsup_{x\to\infty} \frac{1}{\varphi(x)} \int_1^x \frac{\varphi(t)}{t} dt < \infty \tag{12}$$

should hold true.

A function $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a *concave Young function* if for all $x \geq 0$ it is defined by

$$\Psi(x) = \int_0^x \psi(t)\, dt,$$

where $\Psi(0) = 0$ and $\psi : (0, \infty) \to (0, \infty)$ is a decreasing right-continuous such that ψ is integrable on every finite interval $(0, x)$. All along we assume that $\Psi(\infty) = \infty$. For more about concave Young-functions see [49], say).

We note that if Ψ is a concave Young-function, then the following assertions are immediate:

1. $b\Psi$ is also a concave Young function for all positive constant b;

2. $\dfrac{\Psi(x)}{x}$ is a decreasing function on the interval $(0, \infty)$.

3. For all $x \in \mathbb{R}_+$ and every constant $c \in (1, \infty)$ we have

$$\Psi(cx) \leq c\Psi(x). \tag{13}$$

6.2 Moment inequalities for the maximum cumulative sums

Let $X_1, \ldots, X_n$ be arbitrary random variables. Denote

$$S_j := \sum_{k=1}^{j} X_k \quad (S_0 = 0), \quad M_n := \bigvee_{k=1}^{n} |S_k|, \quad S_{i,\,j} := \sum_{k=i}^{j} X_k \text{ if } 1 \leq i \leq j \leq n.$$

The aim of this section is to extend Billingsley's (cf. [19])l Longnecker and Serfling's (cf. [42]), Móricz' (cf. [50]) and Serfling's (cf. [61]) results to convex and concave Young functions, respectively. In doing so we obtain new bounds in the inequalities involved which are easier to be handled and got an improvement for the maximizing constant in [42], Theorem

Theorem 6.2.1 (Agbeko, [4]) *Let $X_1, \ldots, X_n$ be arbitrary random variables.*
Part A.

i. *If $\Phi \in \Delta_2$ and $G \in \nabla$ such that*

$$E\Phi\left(\left|S_{i,\,j}\right|\right) \leq G\left(g\left(i,\,j\right)\right), \tag{14}$$

for all $1 \leq i \leq j \leq n$, where g is a positive function satisfying either

$$g\left(i,\,j\right) + g\left(j+1,\,k\right) \leq g\left(i,\,k\right) \tag{15}$$

whenever $1 \leq k \leq n$, or

$$\frac{g\left(i,\,j\right)}{g\left(1,\,n\right)} \leq \frac{j-i+1}{n} \tag{16}$$

for all $1 \leq i \leq j \leq n$, then

$$E\Phi\left(M_n\right) \leq 3cG\left(g\left(1,\,n\right)\right), \tag{17}$$

where c satisfies (11).

ii. *If $\Phi \in \Delta_2$ and $G \in \nabla \cap \Delta$ such that (14) holds, then*

$$E\Phi\left(M_n\right) \leq c\left\{G\left(0\right) + 2G\left(g\left(1,\,n\right)\right)\right\}, \tag{18}$$

where c satisfies (11).

Part B. *If $\Phi \in \mathcal{Y}_{\text{conc}}$ and $G \in \nabla$ such that (14) holds, then*

$$E\Phi\left(M_n\right) \leq 5G\left(g\left(1,\,n\right)\right). \tag{19}$$

Following the approach of [42] Theorem 6.2.1 can be easily derived from two lemmas below.

Lemma 6.2.1 *Let the non-negative function $g\left(i,\,j\right)$ satisfy either (15) or (16). Then there exist non-negative constants $u_1, \ldots, u_n$ such that*

$$g\left(1,\,n\right) = \sum_{k=1}^{n} u_k \tag{20}$$

and whenever $1 \leq i \leq j \leq n$,

$$g\left(i,\,j\right) \leq \sum_{k=i}^{j} u_k. \tag{21}$$

For the proof, see [42].

Lemma 6.2.2 *Let $X_1, \ldots, X_n$ be arbitrary random variables, Φ and G be any functions.*
Part A.

i.) *If $\Phi \in \Delta_2$ and $G \in \nabla$ such that*

$$E\Phi\left(|S_{i,\,j}|\right) \le G\left(\sum_{k=i}^{j} u_k\right), \tag{22}$$

for all $1 \le i \le j \le n$, then

$$E\Phi\left(M_n\right) \le 3cG\left(\sum_{k=1}^{n} u_k\right), \tag{23}$$

where c satisfies (11).

ii. *If $\Phi \in \Delta_2$ and $G \in \nabla \cap \Delta$ are such that (22) holds, then*

$$E\Phi\left(M_n\right) \le c\left\{G\left(0\right) + 2G\left(\sum_{k=1}^{n} u_k\right)\right\}, \tag{24}$$

where c satisfies (11).

Part B. *If $\Phi \in \mathcal{Y}_{\text{conc}}$ and $G \in \nabla$ are such that (22) holds, then*

$$E\Phi\left(M_n\right) \le 5G\left(\sum_{k=1}^{n} u_k\right). \tag{25}$$

Proof. The proof is similar to that of *Lemma 2* in [42]. In fact, the lemma is obvious for the case $n = 1$. Make the induction hypothesis that the lemma has been established for all counting numbers n satisfying $1 \le n \le N$. Let us show that the lemma is also valid for $n = N$. To this end put $u = u_1 + \ldots + u_N$ and define the integer $m := \min\left\{k : \frac{u}{2} \le u_1 + \ldots + u_k\right\}$. Then

$$\begin{cases} u_1 + \ldots + u_{m-1} \le \frac{u}{2}, \\ u_{m+1} + \ldots + u_N \le \frac{u}{2}. \end{cases} \tag{26}$$

Define

$$L_1 := M_{m-1} = \bigvee_{k=0}^{m-1} |S_k| \quad \text{and} \quad L_2 := \bigvee_{k=m}^{N} |S_k - S_m|.$$

We obviously have that on the one hand $|S_n| \le L_1$ for $1 \le n \le m-1$, and $|S_n| \le |S_m| + L_2$ for $m \le n \le N$ on the other.

Part A/(i). It is obvious that $\Phi\left(M_N\right) \le \Phi\left(L_1\right) + \Phi\left(L_2 + |S_m|\right)$. Inequality (11) implies that

$$\Phi\left(M_N\right) \le \Phi\left(L_1\right) + c\left[\Phi\left(L_2\right) + \Phi\left(|S_m|\right)\right].$$

Thus,

$$E\Phi\left(M_N\right) \le (1 \vee c)\left\{E\Phi\left(L_1\right) + E\Phi\left(L_2\right) + E\Phi\left(|S_m|\right)\right\}. \tag{27}$$

By (22) and (26), the induction hypothesis and the monotonicity of G, we obtain that

$$E\Phi\left(M_N\right) \leq 3\left(1 \vee c\right) G\left(u\right). \tag{28}$$

Similarly, applying twice inequality (11) we observe that

$$\begin{aligned} \Phi\left(M_N\right) &\leq \Phi\left(L_1 + L_2 + |S_m|\right) \leq c\left\{\Phi\left(L_1\right) + \Phi\left(L_2 + |S_m|\right)\right\} \\ &\leq \left(c \vee c^2\right)\left\{\Phi\left(L_1\right) + \Phi\left(L_2\right) + \Phi\left(|S_m|\right)\right\}. \end{aligned} \tag{29}$$

Thus, recalling (22), (26), the induction hypothesis and the monotonicity of function G, we have that

$$E\Phi\left(M_N\right) \leq 3\left(c \vee c^2\right) G\left(u\right). \tag{30}$$

Combining (28) and (30) yields $E\Phi\left(M_N\right) \leq 3\left[\left(1 \vee c\right) \wedge \left(c \vee c^2\right)\right] G\left(u\right)$. Hence $E\Phi\left(M_N\right) \leq 3cG\left(u\right)$, where c satisfies inequality (11). This completes the proof of **Part A/(i)**.

The proof of **Part A/(ii)** is similar to that of **Part A/(i)**. Applying inequalities (22) and (26) in (27) and (29), the induction hypothesis and the convexity of function G yields (24).

Part B. It is obvious by (13) that

$$\begin{aligned} \Phi\left(M_N\right) &\leq \Phi\left(L_1\right) + \Phi\left(L_2 + |S_m|\right) \leq \Phi\left(L_1\right) + \Phi\left(2\left(L_2 \vee |S_m|\right)\right) \\ &\leq \Phi\left(L_1\right) + 2\Phi\left(L_2 \vee |S_m|\right). \end{aligned}$$

But since

$$E\Phi\left(L_2 \vee |S_m|\right) = \int_{(L_2 \geq |S_m|)} \Phi\left(L_2\right) dP + \int_{(L_2 < |S_m|)} \Phi\left(|S_m|\right) dP \leq E\Phi\left(L_2\right) + E\Phi\left(|S_m|\right)$$

it follows that

$$E\Phi\left(M_N\right) \leq E\Phi\left(L_1\right) + 2\left\{E\Phi\left(L_2\right) + E\Phi\left(|S_m|\right)\right\}.$$

Once again applying (22) and (26), the induction hypothesis and the monotonicity of function G in (30), we obtain the desired result in (25). We can thus conclude on the validity of the lemma. ■

Corollary 6.2.1 *Assume that the conditions of Theorem 6.2.1 hold. Then for all positive numbers x,*

$$P\left(M_n \geq x\right) \leq \frac{K}{\Phi\left(x\right)} G\left(g\left(1, n\right)\right),$$

where

$$K := \begin{cases} 3c & \text{if } \Phi \in \Delta_2,\ G \in \nabla \\ 2c & \text{if } \Phi \in \Delta_2,\ G \in \nabla \cap \Delta,\ G\left(0\right) = 0 \\ 5 & \text{if } \Phi \in \mathcal{Y}_{\text{conc}},\ G \in \nabla. \end{cases}$$

Corollary 6.2.2 *Assume that the conditions of Lemma 6.2.2 hold.*

i. *If* $\Phi(x) = x^{\nu}$, $\nu \geq 1$, *and* $G(x) = x^{\gamma}$, $\gamma > 0$, *then* $EM_n^{\nu} \leq 2^{\nu+1}\left(\sum_{k=1}^{n} u_k\right)^{\gamma}$.

ii. *If* $\Phi(x) = x^{\nu}$, $0 < \nu \leq 1$, *and* $G(x) = x^{\gamma}$, $\gamma > 0$, *then* $EM_n^{\nu} \leq 5\left(\sum_{k=1}^{n} u_k\right)^{\gamma}$.

We shall here below show an application for *Theorem 6.2.1.*

Let $(S_n, \mathcal{F}_n)$, $n \in N$, *be a non-negative martingale in* L^2. *Then for every counting number* $n \in N$, *the inequality*

$$EM_n^2 \leq 8ES_n^2 \tag{31}$$

holds, where $M_m = \bigvee_{k=1}^{m} S_k$.

We point out that inequality (31) is due to Garcia [35]. We shall try to derive it from *Theorem 6.3.1* in the following way. Let $(0, d_1, d_2, \ldots)$ be the difference sequence of the submartingale $(S_n, \mathcal{F}_n)$, i.e. for every $n \in \mathbb{N}$, $d_n = S_n - S_{n-1}$ and $S_n = d_1 + \ldots + d_n$.

For every $1 \leq i < j \leq n$, define $S_{i,j} := d_i + \ldots + d_j$. It is easily seen that for $1 \leq i < j \leq n$,

$$S_{i,j} = S_j - S_{i-1} \quad \text{and} \quad ES_{i-1}S_j = E\left(S_{i-1}E\left(S_j | \mathcal{F}_{i-1}\right)\right) \geq ES_{i-1}^2.$$

Hence $ES_{i,j}^2 \leq ES_j^2 - ES_{i-1}^2$. It can easily be checked that the function g, defined by $g(i, j) := ES_j^2 - ES_{i-1}^2$, satisfies the property that $g(i, j) + g(j+1, k) = g(i, k)$. Thus with the choice of $\Phi(x) = x^2$, **Part A/(ii)** of *Theorem 6.3.1* implies inequality (31). To end the application we note that a sharper form of inequality (31) is given by J. L. Doob, where the constant factor equals only 4.

6.3 Maximal inequalities for non-negative submartingales related with concave Young-functions

Definition 6.3.1 (Agbeko, [2, 3]) *We say that for the concave Young function* Φ *the maximal inequality is valid with some positive constant* K_{Φ} *(depending only on* Φ*) if for an arbitrary non-negative submartingale* $(X_n, \mathcal{F}_n)$, $n \in \mathbb{N}$, *the inequality*

$$E\Phi(X_n^*) \leq K_{\Phi}(1 + EX_n) \tag{32}$$

holds for all $n \in \mathbb{N}$, *with* $X_n^* = \bigvee_{k=1}^{n} X_k$.

Theorem 6.3.1 (Agbeko, [3]) *Let* Φ *be any concave Young function. In order that* Φ *satisfy the above maximal inequality, it is necessary and sufficient that*

$$A_{\Phi} := \int_1^{\infty} \frac{\varphi(t)}{t} dt < \infty. \tag{33}$$

Moreover, if $A_{\Phi} < \infty$, *then* $K_{\Phi} = \max(\Phi(1), A_{\Phi})$.

Before we tackle the proof there is the need to mention the motivation behind this theorem and its corresponding definition. Actually in [46], J. Mogyoródi and T. F. Móri proposed the following definition:

For the convex Young function Φ *the Mogyoródi-Móri maximal inequality is said to be valid if there are some constants* a, $b > 0$ *depending only on* Φ *such that for arbitrary non-negative submartingale* $(X_n, \mathcal{F}_n)$, $n \in \mathbb{N}$, *we have* $E(\Phi(X_n^*)) \leq a + E(\Phi(bX_n))$, $n \in \mathbb{N}$.

Then they went on to provide the set of all convex Young functions enjoying this property. We recall their quite remarkable result as follows:

Let (Φ, Γ) *be a pair of conjugate Young functions. In order that* Φ *satisfy the Mogyoródi-Móri maximal inequality it is necessary and sufficient that the power* q *of* Γ (*cf.* 12) *be finite.*

To obtain similar definition and theorem like theirs for concave Young functions imperative adjustments were necessary as expressed in Definition 6.3.1 and Theorem 6.3.1. Now we can prove the theorem left half way, in the same manner Mogyoródi and Móri demonstrated theirs, we must note.

Proof of Theorem 6.3.1. To prove the sufficiency consider the maximal inequality of Doob

$$xP(X_n^* \geq x) \leq EX_n\chi(X_n^* \geq x),\ x > 0,$$

to be integrated on $[1, \infty)$ with respect to the measure generated by the nondecreasing function

$$\int_1^x \frac{\varphi(t)}{t} dt.$$

By the Fubini theorem it follows that

$$E\int_1^{X_n^* \vee 1} \varphi(x)\, dx \leq EX_n \int_1^{X_n^* \vee 1} \frac{\varphi(t)}{t} dt \leq A_\Phi EX_n,$$

since

$$\int_1^{X_n^* \vee 1} \frac{\varphi(t)}{t} dt \leq A_\Phi.$$

Consequently, $E\Phi(X_n^*) \leq \Phi(1) + A_\Phi EX_n$. By choosing $K_\Phi = \max\{A_\Phi, \Phi(1)\}$ we obtain (32).

To prove the necessity suppose that the maximal inequality holds for ϕ with some constant $K_\Phi > 0$. Let (x_n) be an arbitrary sequence of real numbers such that $x_1 = 1$, $x_n < x_{n+1} < 2x_n$, $\lim_{n\to\infty} x_n = \infty$ and

$$A_\Phi := \int_1^\infty \frac{\varphi(t)}{t} dt = \lim_{n\to\infty} \int_1^{x_n} \frac{\varphi(t)}{t} dt. \tag{34}$$

Consider the probability space $\left(\mathbb{N}, 2^{\mathbb{N}}, P\right)$ with $2^{\mathbb{N}}$ is the power set of $\mathbb{N}$. The probability measure is defined on $\left(\mathbb{N}, 2^{\mathbb{N}}\right)$ as follows:

$$P(\{n\}) = \frac{1}{x_n} - \frac{1}{x_{n+1}},\ n \in \mathbb{N}.$$

Define for every $n \in \mathbb{N}$, the random variable

$$X_n(\omega) = x_n \chi(\omega \geq n), \ \omega \in \mathbb{N},$$

and let $\mathcal{F}_n := \sigma(\{1\}, \ldots, \{n-1\} \ldots \{n, n+1, \ldots\})$ be the minimal σ-algebra generated by the measurable partition noted in the brackets.

It can be easily shown that $(X_n, \mathcal{F}_n)$, $n \in \mathbb{N}$, is a non-negative martingale, and

$$X_n^* = \begin{cases} x_\omega & \text{if } \omega < n \\ x_n & \text{if } \omega \geq n. \end{cases}$$

In virtue of the maximal inequality we have

$$\sum_{k=1}^{n-1} \Phi(x_k) \left(\frac{1}{x_k} - \frac{1}{x_{k+1}} \right) + \frac{\Phi(x_n)}{x_n} \leq 2K_\Phi, \tag{35}$$

since here $EX_n = 1$.

The sum of the left hand side of inequality (34) can be estimated as follows:

$$\sum_{k=1}^{n-1} \Phi(x_k) \left(\frac{1}{x_k} - \frac{1}{x_{k+1}} \right) = \sum_{k=1}^{n-1} \Phi(x_k) \int_{x_k}^{x_{k+1}} \frac{1}{t^2} dt \geq \frac{1}{2} \int_1^{x_n} \frac{\Phi(t)}{t^2} dt =$$
$$= \frac{1}{2} \left\{ \Phi(1) - \frac{\Phi(x_n)}{x_n} + \int_1^{x_n} \frac{\varphi(t)}{t} dt \right\}$$

in virtue of the integration by parts. Consequently, (35) implies

$$\int_1^{x_n} \frac{\varphi(t)}{t} dt + \frac{\Phi(x_n)}{x_n} \leq 4K_\Phi - \Phi(1).$$

Passing to the limit, we observe by (34) that

$$\int_1^{\infty} \frac{\varphi(t)}{t} dt \leq 4K_\Phi - \Phi(1) - \alpha < \infty,$$

where $0 \leq \alpha = \lim_{x \to \infty} \dfrac{\Phi(x)}{x}$. Therefore, the argument is a valid one. ■

We shall formulate here without proof the following two results which can be found in [2]. But first let Φ be any concave Young function and denote

$$\xi(x) := \Phi(x) - x\varphi(x). \tag{36}$$

Theorem 6.3.2 (Agbeko, [2]) *Let Φ be any concave Young function and $\xi(x)$ be defined as in* (36)*. Then for any non-negative supermartingale $(X_n, \mathcal{F}_n)$, $n \in \mathbb{N}$, we have*

1. *the inequality*

$$(1-b) E\Phi(X_n^*) - a \leq E\xi(X_n^*)$$

is valid for some constants $a \geq 0$ and $0 < b < 1$, if and only if

$$\limsup_{x \to \infty} \frac{x\varphi(x)}{\Phi(x)} < 1; \tag{37}$$

2. *if inequality* (37) *holds true, then*

$$E\Phi(X_n^*) \le K_\Phi(1 + E\Phi(X_1))$$

for some constant $K_\Phi > 0$ *depending only on* Φ.

Theorem 6.3.3 (Agbeko, [2]) *Let* Φ *be any concave Young function and* $\xi(x)$ *be defined as in* (36)*. Then for any non-negative submartingale* $(X_n, \mathcal{F}_n)$, $n \in \mathbb{N}$,

1. *the inequality*

$$(1-b)\, E\Phi(X_n^*) - a \le E\xi(X_n^*)$$

is valid for some constants $a \ge 0$ *and* $0 < b < 1$, *if and only if*

$$\limsup_{x\to\infty} \frac{x\varphi(x)}{\Phi(x)} < 1; \tag{38}$$

2. *we have*

$$E\xi(X_n^*) \le C_\Phi(1 + EX_n)$$

for some positive constant C_Φ *depending only on* Φ*;*

3. *if inequality* (38) *holds true, then*

$$E\Phi(X_n^*) \le C_\Phi(1 + EX_n)$$

for some positive constant C_Φ *depending only on* Φ.

We say that a concave Young-function Φ satisfies the *density-level property* if $A_\Phi(\infty) < \infty$. The quantity $A_\Phi(\infty)$ will be referred to as *density-level* and the function $A_\Phi : [1, \infty) \to [0, \infty)$, defined by

$$A_\Phi(x) = \int_1^x \frac{\varphi(t)}{t} dt,$$

will be called *density-level function.*

For instance the concave Young-functions $\Phi_1(x) = \sqrt{x}$ and $\Phi_2(x) = \ln(x+1)$, defined for $x \in [0, \infty)$, have finite density-levels. The concave Young-function $\Phi_3(x) = 2x+1-e^{-x}$ is of infinite density-level. In fact, if we let $\varphi_3(x)$ stand for the derivative of function $\Phi_3(x)$, then

$$A_{\Phi_3}(\infty) = \int_1^\infty \frac{\varphi_3(t)}{t} dt \ge \int_1^\infty \frac{2}{t} dt = \infty.$$

Theorem B suggests that the set of concave Young-functions that satisfy the density-level property is a rather broad class.

Define function $A_\Phi^* : (0, \infty) \to (0, \infty]$ by

$$A_\Phi^*(b) = \int_b^\infty \frac{\varphi(x)}{x} dx,$$

where $\Phi \in \mathcal{Y}_{\text{conc}}$.

It is not difficult to see that $A_{\Phi_1}^*(b) < \infty$ and $A_{\Phi_3}^*(b) = \infty$, for any number $b \in (0, \infty)$, where functions $\Phi_1(x) = \sqrt{x}$ and $\Phi_3(x) = 2x + 1 - e^{-x}$ are defined for $x \in [0, \infty)$.

Remark 6.3.1 *The function* $x^{-1}\Phi(x)$ *is decreasing on the interval* $(0, \infty)$ *and*

$$0 \leq \lim_{x\to\infty} \frac{\Phi(x)}{x} < \infty.$$

$\mathfrak{A}$ will denote the set of all functions $\Phi \in \mathcal{Y}_{\text{conc}}$ that satisfy the density-level property.

We note that $\mathfrak{A}$ is a proper subset of $\mathcal{Y}_{\text{conc}}$, since the concave Young-function $\Phi_3 : [0, \infty) \to [0, \infty)$, defined above by $\Phi_3(x) = 2x + 1 - e^{-x}$, was shown to be of infinite density-level.

In the next two sections we study, among others, the closure of $\mathfrak{A}$ under the composition operation. In a sense, Theorems 6.4.1 and 6.4.2 show that the concave Young-functions with the density-level property behave like left and right ideal with respect to the composition operation. We also realized that not every function $\Phi \in \mathcal{Y}_{\text{conc}}$ admits a positive fixed point. The investigation in this direction leads us to isolate a class of functions in $\mathcal{Y}_{\text{conc}}$ enjoying this property. The notion of *degree of contraction* is introduced. We show that every concave Young-function is square integrable with respect to a specific given Lebesgue measure, and we prove that the natural distance defined by the L^2-norm satisfies the metric axioms in $\mathcal{Y}_{\text{conc}}$. We then demonstrate that the subset $\mathfrak{A}$ proves to be a dense set in $\mathcal{Y}_{\text{conc}}$.

6.4 The closure of $\mathfrak{A}$ under addition and composition operations

Remark 6.4.1 *For every number* $s \in (0, \infty)$ *we have that* $s\varphi(s) < \Phi(s)$.

In fact, fix arbitrarily two numbers $s \in (0, \infty)$ and $b \in (0, s)$. Then by applying twice the fact that φ decreases on $(0, \infty)$, we have that

$$\Phi(s) = \int_0^s \varphi(t)\,dt = \int_0^b \varphi(t)\,dt + \int_b^s \varphi(t)\,dt \geq b\varphi(b) + (s-b)\varphi(s) >$$
$$> b\varphi(s) + (s-b)\varphi(s) = s\varphi(s).$$

This proved the remark.

The following remark is an immediate consequence of Theorem B.

Remark 6.4.2 *Let* $\Phi \in \mathcal{Y}_{\text{conc}}$. *If* $\Phi \in \mathfrak{A}$, *then* $\Phi(x) \leq K_\Phi(1+x)$ *for all* $x \in (0, \infty)$, *where* $K_\Phi = \max(\Phi(1), A_\Phi(\infty))$.

Remark 6.4.3 *The composition of two concave Young-functions is also a concave Young-function.*

The following two lemmas are trivial.

Lemma 6.4.1 *For any number* $b \in (0, \infty)$ *and function* $\Phi \in \mathcal{Y}_{\text{conc}}$, *we have that* $b\Phi \in \mathfrak{A}$ *if and only if* $\Phi \in \mathfrak{A}$. *Moreover,* $A_{b\Phi}(x) = bA_\Phi(x)$, $x \in [1, \infty)$.

Lemma 6.4.2 *Let functions* Φ_1 *and* $\Phi_2 \in \mathcal{Y}_{\text{conc}}$ *be arbitrary. Then* Φ_1 *and* $\Phi_2 \in \mathfrak{A}$ *if and only if* $\Phi_1 + \Phi_2 \in \mathfrak{A}$. *Furthermore,* $A_{\Phi_1+\Phi_2}(x) = A_{\Phi_1}(x) + A_{\Phi_2}(x)$, $x \in [1, \infty)$.

Theorem 6.4.1 (Agbeko, [9]) *Let functions* Φ_1 *and* $\Phi_2 \in \mathcal{Y}_{\text{conc}}$ *be arbitrary. If* $\Phi_2 \in \mathfrak{A}$, *then* $\Phi_1 \circ \Phi_2 \in \mathfrak{A}$.

Proof. Write φ_i for the derivative of Φ_i ($i \in \{1, 2\}$). Compute the density-level of the composition $\Phi_1 \circ \Phi_2$.

$$\begin{aligned} A_{\Phi_1 \circ \Phi_2}(\infty) &= \int_1^\infty \frac{\varphi_2(x)\,\varphi_1(\Phi_2(x))}{x} dx \\ &\le \varphi_1(\Phi_2(1)) \int_1^\infty \frac{\varphi_2(x)}{x} dx = \varphi_1(\Phi_2(1))\, A_{\Phi_2}(\infty) < \infty, \end{aligned}$$

via the monotonicity of function φ_1. ■

Remark 6.4.4 *Let* $\Phi \in \mathcal{Y}_{\text{conc}}$. *Then for* Φ *to belong to* $\mathfrak{A}$ *it is necessary that* $\lim_{t\to\infty} \varphi(t) = 0$.

Proof. Assume that $\Phi \in \mathfrak{A}$ but $\lim_{t\to\infty} \varphi(t) = l_0 > 0$. Pick an arbitrarily fixed number $t \in (1, \infty)$. Then

$$\infty > A_\Phi(\infty) \ge \int_1^t \frac{\varphi(x)}{x} dx \ge \varphi(t) \log(t) > l_0 \log(t).$$

Passing to the limit, it will follow that $\infty = A_\Phi(\infty) < \infty$, which is absurd. This completes the proof. ■

The following remark suggests that if $\Phi \in \mathcal{Y}_{\text{conc}}$, then either $A^*_\Phi(b) = \infty$ for all $b \in (0, \infty)$, or $A^*_\Phi(b) < \infty$ for all $b \in (0, \infty)$.

Remark 6.4.5 *Let* $\Phi \in \mathcal{Y}_{\text{conc}}$. *Then* $A^*_\Phi(b) < \infty$ *for every constant* $b \in (0, \infty) \setminus \{1\}$ *if and only if* $A_\Phi(\infty) < \infty$.

Proof. A simple computation shows that

$$A^*_\Phi(b) = \int_b^\infty \frac{\varphi(x)}{x} dx = \begin{cases} A_\Phi(\infty) + \int_b^1 \frac{\varphi(x)}{x} dx & \text{if } b < 1 \\ \\ A_\Phi(\infty) - \int_1^b \frac{\varphi(x)}{x} dx & \text{if } b > 1 \end{cases}$$

which yields the result. ■

Theorem 6.4.2 (Agbeko, [9]) *Let functions* Φ_1 *and* $\Phi_2 \in \mathcal{Y}_{\text{conc}}$ *be arbitrary. If* $\Phi_1 \in \mathfrak{A}$, *then* $\Phi_1 \circ \Phi_2 \in \mathfrak{A}$.

Proof. We first show that

$$A_{\Phi_1}(\infty) = \int_{\Phi_2^{-1}(1)}^{\infty} \frac{\varphi_2(t)\,\varphi_1(\Phi_2(t))}{\Phi_2(t)} dt,$$

where Φ_2^{-1} is the inverse function of Φ_2 (whose existence is guaranteed by the continuity of Φ_2).

In fact, by definition we have that

$$A_{\Phi_1}(\infty) = \int_1^{\infty} \frac{\varphi_1(x)}{x} dx.$$

Now, setting $x = \Phi_2(t)$ we observe that $dx = \varphi_2(t)\,dt$ and thus

$$A_{\Phi_1}(\infty) = \int_{\Phi_2^{-1}(1)}^{\infty} \frac{\varphi_2(t)\,\varphi_1(\Phi_2(t))}{\Phi_2(t)} dt.$$

Next, compute the density-level of the composition $\Phi_1 \circ \Phi_2$. Remark 6.3.1 implies that

$$\begin{aligned} A_{\Phi_1 \circ \Phi_2}(\infty) &= \int_1^{\infty} \frac{\varphi_2(t)\,\varphi_1(\Phi_2(t))}{t} dt \\ &= \int_1^{\infty} \frac{\Phi_2(t)}{t} \frac{\varphi_2(t)\,\varphi_1(\Phi_2(t))}{\Phi_2(t)} dt \\ &\leq c \int_{\Phi_2^{-1}(1)}^{\infty} \frac{\varphi_2(t)\,\varphi_1(\Phi_2(t))}{\Phi_2(t)} dt = c A_{\Phi_1}(\infty), \end{aligned}$$

where $c = \frac{1}{\Phi_2^{-1}(1)}$ (the second equality holds because of the claim shown above), which was to be proven. ■

Corollary 6.4.1 *Let $\Phi \in \mathcal{Y}_{\text{conc}}$ and $\alpha \in (0, 1)$ be arbitrary. Then $\Phi_\alpha \in \mathfrak{A}$, where the function $\Phi_\alpha : [0, \infty) \to [0, \infty)$ is defined by $\Phi_\alpha(x) = \Phi^\alpha(x) = (\Phi(x))^\alpha$.*

Proposition 6.4.1 *Let $x, y \in (0, \infty)$ and $\Delta \subset \mathcal{Y}_{\text{conc}}$ (with $\Delta \neq \varnothing$) be arbitrary. Then*

$$\left| \sup_{\Phi \in \Delta} \Phi(x) - \sup_{\Phi \in \Delta} \Phi(y) \right| \leq \sup_{\Phi \in \Delta} |\Phi(x) - \Phi(y)|,$$

provided that $\sup_{\Phi \in \Delta} \Phi(t) < \infty$ *for all* $t \in (0, \infty)$.

Proof. We first note that

$$\Phi(x) \leq |\Phi(x) - \Phi(y)| + \Phi(y) \quad \text{and} \quad \Phi(y) \leq |\Phi(x) - \Phi(y)| + \Phi(x).$$

Taking the supremum we can easily observe that

$$\sup_{\Phi \in \Delta} \Phi(x) \leq \sup_{\Phi \in \Delta} |\Phi(x) - \Phi(y)| + \sup_{\Phi \in \Delta} \Phi(y)$$

and

$$\sup_{\Phi\in\Delta}\Phi(y)\leq\sup_{\Phi\in\Delta}|\Phi(x)-\Phi(y)|+\sup_{\Phi\in\Delta}\Phi(x).$$

Combining these inequalities we have that

$$-\sup_{\Phi\in\Delta}|\Phi(x)-\Phi(y)|\leq\sup_{\Phi\in\Delta}\Phi(x)-\sup_{\Phi\in\Delta}\Phi(y)\leq\sup_{\Phi\in\Delta}|\Phi(x)-\Phi(y)|,$$

which yields the result. ■

We know that $k\Phi\in\mathcal{Y}_{\text{conc}}$ for any fixed $\Phi\in\mathcal{Y}_{\text{conc}}$ and all $k\geq 1$. Then

$$\sup_{\Phi\in\mathcal{Y}_{\text{conc}}}\Phi(x)\geq\sup_{k\geq 1}k\Phi(x)=\begin{cases}0 & \text{if } x=0\\ \infty & \text{if } x\in(0,\infty),\end{cases}$$

meaning that there is no real function $g(x)$ such that $\Phi(x)\leq g(x)$ for all $\Phi\in\mathcal{Y}_{\text{conc}}$ and $x\in[0,\infty)$. Nevertheless, this is possible for their suitably normalized forms, as shown in the following lemma.

Lemma 6.4.3 *The function* $S:[0,\infty)\rightarrow[0,\infty)$, *defined by*

$$S(x)=\sup_{\Phi\in\mathcal{Y}_{\text{conc}}}(\Phi(1))^{-1}\Phi(x),$$

enjoys the following properties:

1. $S(0)=0$ *and* $S(1)=1$.
2. S *is a non-decreasing function such that* $(\Phi(1))^{-1}\Phi(x)\leq S(x)$ *for all* $\Phi\in\mathcal{Y}_{\text{conc}}$ *and* $x\in[0,\infty)$.
3. *The identity* $\sup_{\Phi\in\mathcal{Y}_{\text{conc}}}(1+\Phi(1))^{-1}=1$ *holds.*
4. *For every number* $x\in[0,\infty)$, *the chain of inequalities* $x\leq S(x)\leq x+1$ *holds true.*
5. *We have that* $\lim_{x\rightarrow\infty}\dfrac{S(x)}{x}=1$ *and* $\lim_{x\rightarrow\infty}S(x)=\infty$.

Proof. The first part is obvious. We show that $S(x)$ is a non-decreasing function. In fact, pick arbitrarily two numbers $x_1$and $x_2\in[0,\infty)$ with $x_1<x_2$. By the monotonicity we have that $\Phi(x_1)<\Phi(x_2)$. If we normalize this inequality suitably and then take the supremum on both sides over all $\Phi\in\mathcal{Y}_{\text{conc}}$ we can then observe that $S(x_1)\leq S(x_2)$. Thus S is a non-decreasing function. To show the identity in the third part we begin by establishing the inequality $(1+\Phi(1))^{-1}\leq 1$, which holds for every $\Phi\in\mathcal{Y}_{\text{conc}}$. Then $\sup_{\Phi\in\mathcal{Y}_{\text{conc}}}(1+\Phi(1))^{-1}\leq 1$. We also know that $k^{-1}\Phi\in\mathcal{Y}_{\text{conc}}$ for any fixed integer $k\geq 1$. Hence $(1+k^{-1}\Phi(1))^{-1}\leq\sup_{\Phi\in\mathcal{Y}_{\text{conc}}}(1+\Phi(1))^{-1}$. Passing to the limit we observe that $\lim_{k\rightarrow\infty}(1+k^{-1}\Phi(1))^{-1}=1$. Consequently $\sup_{\Phi\in\mathcal{Y}_{conc}}(1+\Phi(1))^{-1}=1$. The fourth part will be proved if we show that $S(x)\leq x+1$ and $S(x)\geq x$ for all $x\in[0,\infty)$. In fact, take arbitrarily a function $\Phi\in\mathcal{Y}_{\text{conc}}$. Clearly the equation of the tangent line of Φ at the point $(1,\ \Phi(1))$ is given by $y=\varphi(1)(x-1)+\Phi(1)$, $x\in[0,\infty)$. Via the concavity of Φ, it is obvious that $\Phi(x) \leq \varphi(1)(x-1) + \Phi(1)$, $x \in [0, \infty)$. Hence by *Remark 6.4.1* we have: $\Phi(x) \leq \varphi(1)x + \Phi(1) < \Phi(1)(x+1)$, $x \in [0, \infty)$. This implies that $S(x) < x+1$, for all $x \in [0, \infty)$. Finally fix arbitrarily a function $\Phi \in \mathcal{Y}_{\text{conc}}$. Then the function, defined on $[0, \infty)$ by $x + \Phi(x)$ (for any fixed $\Phi \in \mathcal{Y}_{\text{conc}}$), also belongs to $\mathcal{Y}_{\text{conc}}$. Hence

$$S(x) \geq \frac{x + \Phi(x)}{1 + \Phi(1)} \geq \frac{x}{1 + \Phi(1)},\ x \in [0, \infty).$$

Now taking the supremum over $\Phi \in \mathcal{Y}_{\text{conc}}$, the third part leads to the desired inequality $S(x) \geq x$. To complete the proof we just point out that the fifth part grows obvious because of the fourth part. ■

6.5 The fixed points of a class of concave Young-functions

In mathematical analysis, there are various fixed-point theorems. Fixed points are also known as equilibria or stationary points. We shall remind only three of these theorems. The *Kakutani* fixed point theorem is a fixed-point theorem for set-valued functions. It provides sufficient conditions for a set-valued function defined on a convex, compact subset of an Euclidean space $\mathbb{R}^n$ to have a fixed point, i.e. a point which is mapped to a set containing the point itself.

The *Kakutani*'s fixed point theorem developed in 1941 [38] was famously used by *John Nash* [51] in his description of *Nash equilibrium*, which has been worth him the Nobel Price in Economics. It has subsequently found widespread applications in game theory besides the economics application.

The Kakutani fixed point theorem is actually a generalization of *Brouwer* fixed point theorem [20]. It states that every continuous function from the closed unit ball D^n to itself has at least one fixed point (where n is any positive integer, the closed unit ball is the set of all points in Euclidean $\mathbb{R}^n$ which are at distance at most 1 from the origin, and a fixed point of a function $f : D^n \to D^n$ is a point x in D^n such that $f(x) = x$).

The Brouwer fixed point theorem is a fundamental result in topology which proves the existence of fixed points for continuous functions defined on compact, convex subsets of Euclidean spaces.

Perhaps it is worth mentioning some two elegant illustrations of the Brouwer fixed point theorem. (See
http://www.marginalrevolution.com/marginalrevolution/2004/08/kakutani_is_at_.html)

1. "*One morning, exactly at sunrise, a Buddhist monk began to climb a tall mountain. The narrow path, no more than a foot or two wide, spiraled around the mountain to a glittering temple at the summit. The monk ascended the path at varying rates of speed, stopping many times along the way to rest and to eat the dried fruit he carried with him. He reached the temple shortly before sunset. After several days of fasting and meditation he began his journey back along the same path, starting at sunrise and again walking at variable speeds with many pauses along the way. His average speed descending was, of course, greater than his average climbing speed. Prove that* *there is a spot along the path that the monk will occupy on both trips at precisely the same time of day.*"

"Here is an intuitive proof of the monk problem. Imagine that there are two monks, one going down and one going up, each beginning on the same day at sunrise. At some point in the day the hiker's must meet!" However, we must note that the Brouwer's fixed point theorem guarantees a rigorous existence of such spot.

2. *Take two sheets of graph paper of equal size with coordinate systems on them, lay one flat on the table and crumple up (without ripping or tearing) the other one and place it any fashion on top of the first so that the crumpled paper does not reach outside the flat one. There will then be at least one point of the crumpled sheet that lies exactly on top of its corresponding point* (i.e. *the point with the same coordinates*) *of the flat sheet.*

This is a consequence of the $n = 2$ case of Brouwer's theorem applied to the continuous map that assigns to the coordinates of every point of the crumpled sheet the coordinates of the point of the flat sheet immediately beneath it.

In dynamic models, stationary equilibrium is typically described as a solution of the equation $x = f(x)$, where f is a mapping which determines the current state as a function of the previous state, or as a function of the expected future state. In many cases x is a finite dimensional vector, and in general positive solutions (i.e. fixed points of f) are rather sought for. We note that this type of problem have been investigated for decades, and in many occasions for concave functions.

The third well-known fixed point theorem, though more restrictive than the Brouwer's one, guarantees the existence of a unique fixed point. We remind the following definition:

Definition 6.5.1 *A function T from a metric space (M, ρ) to itself is called a contraction if there is an α which satisfies $0 \leq \alpha < 1$ so that $\rho(T(x), T(y)) \leq \alpha\rho(x, y)$ for all $x, y \in M$.*

Contraction Mapping Principle ([57]) *Let T be a contraction on a complete metric space (M, ρ). Then there is a unique point $x \in M$ (called fixed point) such that $T(x) = x$. Furthermore, if x_0 is any point in M and we define $x_{n+1} = T(x_n)$, then $\lim_{n\to\infty} x_n = x$.*

I should like to mention the nice work of my colleague J. Mészáros (cf. [43]) in which he connected various forms of contraction principles.

We shall seek for all those positive numbers which can be a fixed point for any given concave Young function.

Theorem 6.5.1 *Let $\Phi \in \mathcal{Y}_{\text{conc}}$ and c^* be any positive number. In order that the equality $\Phi(c^*) = c^*$ hold, it is necessary and sufficient that the range of the function $\Phi|_{[c^*,\infty)} : [c^*, \infty) \to [0, \infty)$, defined by $\Phi|_{[c^*,\infty)}(x) = \Phi(x)$, should equal the interval $[c^*, \infty)$.*

Proof. Suppose that $\Phi|_{[c^*,\infty)}(c^*) = \Phi(c^*) = c^*$. Obviously Φ is a bijection on $[0, \infty)$. Hence it follows that $\Phi|_{[c^*,\infty)}$ is an injection on $[c^*, \infty)$. Since $\Phi|_{[c^*,\infty)}$ is continuous on $[c^*, \infty)$ and tends increasingly to ∞, we have that the range of function $\Phi|_{[c^*,\infty)}$ equals $[\Phi(c^*), \infty) = [c^*, \infty)$, by assumption. Conversely, assume that the range of $\Phi|_{[c^*,\infty)}$ equals interval $[c^*, \infty)$, but in the contrary there is some number $y \in (c^*, \infty)$ such that $\Phi(y) = \Phi|_{[c^*,\infty)}(y) = c^*$. By the assumption it is obvious that function $\Phi|_{[c^*,\infty)}$ is surjective on $[c^*, \infty)$. Moreover, $\Phi|_{[c^*,\infty)}$ maps bijectively the interval $[c^*, \infty)$ onto itself because it is also an injection. The monotonicity of Φ yields that $\Phi(c^*) = \Phi|_{[c^*,\infty)}(c^*) < \Phi(y) = c^*$. However, by the bijective property of $\Phi|_{[c^*,\infty)}$, we have that $\Phi(c^*) \geq c^*$. Consequently the inequality $c^* < c^*$ will follow. This, however, is absurd. Therefore, we can conclude on the validity of the argument. ■

Proposition 6.5.1 (Agbeko, [9]) *Let $\Phi \in \mathcal{Y}_{\mathrm{conc}}$ be arbitrary and fix any number $s \in (0, \infty)$. Then*

$$|\Phi(x) - \Phi(y)| \leq \varphi(s)\,|x - y|$$

for all numbers $x, y \in [s, \infty)$.

Proof. Pick numbers $x, y \in [s, \infty)$ arbitrarily. Via the monotonicity of Φ it follows that

$$\begin{aligned} |\Phi(x) - \Phi(y)| &= \max(\Phi(x), \Phi(y)) - \min(\Phi(x), \Phi(y)) \\ &= \Phi(\max(x, y)) - \Phi(\min(x, y)). \end{aligned}$$

Hence the monotonicity of φ yields that

$$|\Phi(x) - \Phi(y)| = \int_{\min(x,y)}^{\max(x,y)} \varphi(t)\,dt \leq \varphi(s)(\max(x, y) - \min(x, y)) = \varphi(s)\,|x - y|.$$

This was to be proved. ■

Next, we characterize the existence of positive fixed points of concave Young-functions according to some behavior of their derivatives.

Theorem 6.5.2 (Agbeko, [11]) *Let $\Phi \in \mathcal{Y}_{\mathrm{conc}}$ be arbitrary. In order that there be a constant $s > 0$ for which $\varphi(s) < 1$, it is necessary and sufficient that Φ admit a positive fixed point, i.e. $\Phi(x) = x$ for some number $x > 0$.*

Proof. To prove the sufficiency assume that there is a number $s > 0$ such that $\varphi(s) < 1$. Then by recalling *Proposition 6.5.1* one can easily observe that Φ is a contraction in the interval (s, ∞). Consequently, the *Contraction Principle* (cf. [57]) yields $\Phi(x) = x$ for some $x \geq s$. Next, let us show the necessity. Assume that there exists some $x_0 > 0$ for which $\Phi(x_0) = x_0$, but in the contrary $\varphi(t) \geq 1$ for all $t > 0$. Then it is easy to check that $\Phi(x) \geq x$ for all $x > 0$. Since Φ is a strictly concave and increasing function, the graph of Φ must lie below that of the line $y = x$ on the interval (x_0, ∞). This fact, however, contradicts that $\Phi(x) \geq x$ for all $x > 0$. Therefore, we can conclude on the validity of the argument. ■

Proposition 6.5.2 *Let $\Phi \in \mathcal{Y}_{\text{conc}}$ be arbitrary. If $x_0 \in (0, \infty)$ is such that $\Phi(x_0) = x_0$, then $\varphi(x_0) < 1$.*

Proof. It is not difficult to see that $\Phi(t) \geq t\varphi(t)$ whenever $t \in (0, \infty)$. Assume the existence of some $x_0 \in (0, \infty)$ for which $\Phi(x_0) = x_0$. Then as noted above

$$x_0 = \Phi(x_0) \geq x_0 \varphi(x_0),$$

and hence $\varphi(x_0) \leq 1$. Now, suppose that $\varphi(x_0) = 1$. Since φ is a decreasing function on $(0, \infty)$, there must be some $\varepsilon \in (0, 1)$ such that $\varphi(x_0 + \varepsilon) < 1$, making Φ be a contraction on $(x_0 + \varepsilon, \infty)$, via *Proposition 6.5.1.* But then it would mean that there must be some $x^* \in (x_0 + \varepsilon, \infty)$ with $\Phi(x^*) = x^*$. Necessarily, it would ensue that Φ is not a concave function on the interval $(x_0, x^*]$, a contradiction, indeed. Therefore, $\varphi(x_0) < 1$. This was to be proven. ■

Definition 6.5.2 *A number $s > 0$ is called the degree of contraction of a function $\Phi \in \mathcal{Y}_{\text{conc}}$ if $\varphi(s) = 1$.*

We note in this case that $\varphi(s + \delta) < 1$ for any positive number δ, which makes Φ a contraction for some suitable δ.

The degree of contraction can provide a starting point for any iteration for finding the positive fixed points of concave Young-functions. In this viewpoint the degree of contraction can be useful, as a matter of fact.

Example 6.5.1 *The degree of contraction of $\Phi(x) = 4\left(\sqrt{x+1} - 1\right)$, $x \in [0, \infty)$, equals* 3.

Example 6.5.2 *For any fixed number $p \in (0, 1)$, the degree of contraction of the function $\Phi_p(x) = x^p$, $x \in [0, \infty)$ is equal to $p^{1/(1-p)}$.*

Example 6.5.3 *The function $\Phi(x) = \log(x+1)$, $x \in [0, \infty)$, has no degree of contraction.*

Example 6.5.4 *The degree of contraction of function $\Phi(x) = 2\log(x+1)$ exists and equals* 1.

Example 6.5.5 *The concave Young function Φ defined by $\Phi(x) = \frac{x}{2} + \sqrt{x}$ does not meet condition* (3)*. Yet its degree of contraction exists and equals* 1.

6.6 Is set $\mathfrak{A}$ dense in $\mathcal{Y}_{\text{conc}}$?

We shall answer this question in the affirmative.

Theorem 6.6.1 (Agbeko, [9]) *For any concave Young-function Φ, there exists a sequence $(\Phi_n) \subset \mathfrak{A}$ such that (Φ_n) converges pointwise to Φ, i.e. $\lim_{n\to\infty} \Phi_n(x) = \Phi(x)$ whenever $x \in [0, \infty)$.*

Proof. Fix arbitrarily an index $n \geq 1$ and define $\Phi_n(x) = \Phi^{n/(n+1)}(x)$, $x \in [0, \infty)$. Obviously $(\Phi_n) \subset \mathcal{Y}_{\text{conc}}$ because of Remark 6.4.3. So, on the one hand, Corollary 6.4.1 yields that $(\Phi_n) \subset \mathfrak{A}$. On the other hand we can easily see in the limit that

$$\lim_{n\to\infty} \Phi_n(x) = \lim_{n\to\infty} \Phi^{n/(n+1)}(x) = \Phi(x)$$

for every $x \in [0, \infty)$. Therefore, we conclude on the validity of the theorem. ■

Lemma 6.6.1 *Let $\Phi \in \mathcal{Y}_{\text{conc}}$. Then there are constants $C_\Phi > 0$ and $B_\Phi \geq 0$ such that*

$$A_\Phi(\infty) - B_\Phi \leq \int_0^\infty \frac{\Phi(t)}{(t+1)^2} dt \leq C_\Phi + A_\Phi(\infty).$$

Proof. An integration by parts leads

$$\int_0^\infty \frac{\Phi(t)}{(t+1)^2} dt = \left[\frac{-\Phi(t)}{t+1}\right]_0^\infty + \int_0^\infty \frac{\varphi(t)}{t+1} dt = \int_0^\infty \frac{\varphi(t)}{t+1} dt - B_\Phi, \tag{39}$$

where $0 \leq B_\Phi := \lim_{t\to\infty} \dfrac{\Phi(t)}{t+1} < \infty$, as $\dfrac{\Phi(t)}{t+1} < \dfrac{\Phi(t)}{t}$ for all $t \in (0, \infty)$. On the one hand,

$$\int_0^\infty \frac{\varphi(t)}{t+1} dt = \int_0^1 \frac{\varphi(t)}{t+1} dt + \int_1^\infty \frac{\varphi(t)}{t+1} dt \leq \int_0^1 \frac{\varphi(t)}{t+1} dt + A_\Phi(\infty). \tag{40}$$

On the other hand, by the monotonicity of function $\varphi(t)$ and by the change of variables, we have that

$$\int_0^\infty \frac{\varphi(t)}{t+1} dt \geq \int_0^\infty \frac{\varphi(t+1)}{t+1} dt = \int_1^\infty \frac{\varphi(x)}{x} dx = A_\Phi(\infty). \tag{41}$$

Consequently, if we combine (39)–(41), one can observe that

$$A_\Phi(\infty) - B_\Phi \leq \int_0^\infty \frac{\Phi(t)}{(t+1)^2} dt \leq \int_0^1 \frac{\varphi(t)}{t+1} dt + B_\Phi + A_\Phi(\infty).$$

This leads to the desired result. ■

Lemma 6.6.1 suggests that the quantity $\int_0^\infty \frac{\Phi(t)}{(t+1)^2} dt$ and the density-level $A_\Phi(\infty)$ are equivalent, in the sense that they are both either finite or infinite. This gives rise to the following essential result.

Lemma 6.6.2 *Let $\Phi \in \mathcal{Y}_{\text{conc}}$ be arbitrary. Then*

$$\int_0^\infty \frac{(\Phi(x))^2}{(x+1)^4} dx < \infty.$$

Proof. Clearly,

$$\int_0^\infty \frac{(\Phi(x))^2}{(x+1)^4}dx = \int_0^1 \frac{(\Phi(x))^2}{(x+1)^4}dx + \int_1^\infty \frac{(\Phi(x))^2}{(x+1)^4}dx$$
$$\leq \int_0^1 \frac{(\Phi(x))^2}{(x+1)^4}dx + \int_1^\infty \frac{(\Phi(x))^2}{x^4}dx.$$

Integration by parts yields that

$$\int_1^\infty \frac{(\Phi(x))^2}{x^4}dx = \frac{\Phi(1)}{3} + \frac{2}{3}\int_1^\infty \frac{\varphi(x)\Phi(x)}{x^3}dx \leq \frac{\Phi(1)}{3} + \frac{2\varphi(1)\Phi(1)}{3},$$

because $\varphi(x)$ and $\dfrac{\Phi(x)}{x}$ are decreasing functions. ■

Now endow the half line $[0, \infty)$ with a σ-algebra $\mathcal{M}$ containing the Borel sets. Define a Lebesgue measure $\mu : \mathcal{M} \to [0, \infty)$ by $\mu([0, x)) = \dfrac{1}{3}\left(1 - \dfrac{1}{(x+1)^3}\right)$ for all $x \in [0, \infty)$. Let $L^2 := L^2([0, \infty), \mathcal{M}, \mu)$ be the collection of all (measurable) square integrable functions. We know (see [60], Remark 11.37, page 326) that the pair (L^2, d) is not a metric space unless we identify functions which differ only on a set of measure zero, where the mapping $d : L^2 \times L^2 \to [0, \infty)$ is defined by

$$d(f, g) = \sqrt{\int_{[0,\infty)} (f-g)^2 \, d\mu} = \sqrt{\int_0^\infty \frac{(f(x) - g(x))^2}{(x+1)^4}dx}.$$

By Lemma 6.6.2, we observe that $\mathcal{Y}_{\text{conc}} \subset L^2$. Unfortunately we note that this does not guarantee that the pair $(\mathcal{Y}_{\text{conc}}, d)$ is a metric space, for the reason mentioned above. Nevertheless, we shall prepare the ground for showing that $(\mathcal{Y}_{\text{conc}}, d)$ is actually a metric space.

Whenever $\Phi \in \mathcal{Y}_{\text{conc}}$ write G_Φ for the graph of Φ on $[0, \infty)$, i.e.

$$G_\Phi = \{(x, \Phi(x)) : x \in [0, \infty)\}$$

and write $G_\Phi^{a||b}$ for the graph of Φ on the interval $[a, b)$, i.e.

$$G_\Phi^{a||b} = \{(x, \Phi(x)) : x \in (a, b)\},$$

where $a < b$ are any non-negative numbers.

Lemma 6.6.3 *Let Φ and $\Psi \in \mathcal{Y}_{\text{conc}}$ be arbitrary with distinct graphs. Then*

$$|\{x \in (0, \infty) : \Phi(x) = \Psi(x)\}| \leq 1,$$

where $|B|$ stands for the cardinality of B whenever B is a set.

Proof. Suppose in the contrary that

$$|\{x \in (0, \infty) : \Phi(x) = \Psi(x)\}| \geq 2.$$

Write

$$x_1 = \inf\{x \in (0, \infty) : \Phi(x) = \Psi(x)\}$$

and

$$x_2 = \inf\{x \in (0, \infty) \setminus \{x_1\} : \Phi(x) = \Psi(x)\}.$$

It is clear that $0 < x_1 < x_2$ and $\Phi(x_i) = \Psi(x_i)$, $i \in \{1, 2\}$. We point out that the two graphs are continuous. We show that the graph of one of the functions Φ and Ψ lies above the graph of the other on the interval $(0, x_1)$. In fact, without loss of generality we may assume in the contrary that $G_\Phi^{0||x_1}$ lies both above and below $G_\Psi^{0||x_1}$. Then necessarily the two graphs must cross each other in the interior of interval $(0, x_1)$, i.e. there is some $x_0 \in (0, x_1)$ such that $\Phi(x_0) = \Psi(x_0)$. This, however is in contradiction with the minimality of x_1. Hence we can assume that $G_\Phi^{0||x_1}$ lies above $G_\Psi^{0||x_1}$. By the continuity and the fact that $\Phi(x_1) = \Psi(x_1)$ we note that G_Φ crosses G_Ψ at point $(x_1, \Phi(x_1))$. Nevertheless, since both Φ and Ψ are unbounded increasing functions and $\Phi(x_2) = \Psi(x_2)$, the graph G_Φ must cross the graph G_Ψ at point $(x_2, \Phi(x_2))$. This means that Φ must be convex on the interval (x_1, x_2), which is absurd since these functions are concave. ■

Corollary 6.6.1 *Let Φ and $\Psi \in \mathcal{Y}_{\text{conc}}$ be arbitrary. Then among the following three assertions exactly one fulfills:*

1. $\{x \in [0, \infty) : \Phi(x) = \Psi(x)\} = [0, \infty)$.
2. $\{x \in (0, \infty) : \Phi(x) \neq \Psi(x)\} = (0, \infty)$.
3. *There is a unique number $x^* \in (0, \infty)$ with $\Phi(x^*) = \Psi(x^*)$ such that*

$$\{x \in (0, \infty) \setminus \{x^*\} : \Phi(x) \neq \Psi(x)\} = (0, \infty) \setminus \{x^*\}.$$

Lemma 6.6.4 *Let Φ and $\Psi \in \mathcal{Y}_{\text{conc}}$ be arbitrary. Then in order that $\Phi(x) = \Psi(x)$ for all $x \in [0, \infty)$ it is necessary and sufficient that*

$$\int_{[0,\infty)} (\Phi - \Psi)^2 \, d\mu = 0.$$

Proof. We first note that the sufficiency is obvious. To show the necessity let us assume that

$$\int_{[0,\infty)} (\Phi - \Psi)^2 \, d\mu = 0.$$

Then on the one hand $\mu(\{x \in [0, \infty) : \Phi(x) = \Psi(x)\}) = \mu([0, \infty)) = \frac{1}{3}$ so that necessarily
$\{x \in [0, \infty) : \Phi(x) = \Psi(x)\} \neq \varnothing$. On the other hand

$$\mu(\{x \in (0, \infty) : \Phi(x) \neq \Psi(x)\}) = 0.$$

Note that both the sets $\{x \in [0, \infty) : \Phi(x) = \Psi(x)\}$ and $\{x \in (0, \infty) : \Phi(x) \neq \Psi(x)\}$ cannot be non-empty at the same time (because of Corollary 6.6.1). Consequently,

$$\{x \in (0, \infty) : \Phi(x) \neq \Psi(x)\} = \varnothing$$

and, therefore, $\{x \in [0, \infty) : \Phi(x) = \Psi(x)\} = [0, \infty)$. ■

We are now in the position to state the result hereby.

Proposition 6.6.1 *The mapping $d : \mathcal{Y}_{\text{conc}} \times \mathcal{Y}_{\text{conc}} \to [0, \infty)$, defined by*

$$d(\Phi, \Psi) = \sqrt{\int_{[0, \infty)} (\Phi - \Psi)^2 \, d\mu} = \sqrt{\int_0^\infty \frac{(\Phi(x) - \Psi(x))^2}{(x+1)^4} dx},$$

satisfies the metric axioms, i.e. for any three functions Φ_1, Φ_2 and $\Phi_3 \in \mathcal{Y}_{\text{conc}}$

1. $d(\Phi_1, \Phi_2) \geq 0$ *and* $d(\Phi_1, \Phi_2) = 0$ *if and only if* $\Phi_1 = \Phi_2$.

2. $d(\Phi_1, \Phi_2) = d(\Phi_2, \Phi_1)$.

3. $d(\Phi_1, \Phi_2) \leq d(\Phi_1, \Phi_3) + d(\Phi_3, \Phi_2)$.

The pair $(\mathcal{Y}_{\text{conc}}, d)$ is a metric space and hence by referring to Theorem C the pair $(\mathfrak{A}, d)$ is also a metric space.

Theorem 6.6.2 *Let $\Phi \in \mathcal{Y}_{\text{conc}}$ and write $\Phi_n = \Phi^{n/(n+1)}$, $n \geq 1$. Then*

$$\lim_{n\to\infty} \int_{[0, \infty)} \Phi_n^2 d\mu = \int_{[0, \infty)} \Phi^2 d\mu.$$

Proof. For every index $n \geq 1$, define $\Phi_n^* := (\Phi_n(1))^{-1} \Phi_n$. Clearly, $(\Phi_n) \subset \mathcal{Y}_{\text{conc}}$ (see Corollary 6.4.1) and hence $(\Phi_n^*) \subset \mathfrak{A}$ because of Lemma 6.4.1. Via Theorem 6.6.1 it follows that sequence (Φ_n) converges to Φ pointwise, which in turn entails that sequence (Φ_n^*) converges to $(\Phi(1))^{-1} \Phi$ pointwise. Write the function $Z(x) := x + 1$, $x \in [0, \infty)$. We obtain (by Lemma 6.4.3) that

$$\sup_{n\geq 1} \Phi_n^*(x) \leq S(x) \leq Z(x), \; x \in [0, \infty).$$

Now, on the one hand a simple computation shows that $Z \in L^2$. On the other hand we can deduce from Lemma 6.6.2 that $(\Phi_n) \subset L^2$ and thus $(\Phi_n^*) \subset L^2$. Therefore, the Dominated Convergence Theorem guarantees that

$$\lim_{n\to\infty} \int_{[0, \infty)} \Phi_n^{*2} d\mu = (\Phi(1))^{-2} \int_{[0, \infty)} \Phi^2 d\mu.$$

Now we remark that for every index $n \geq 1$,

$$\int_{[0, \infty)} \Phi_n^2 d\mu = (\Phi(1))^2 \int_{[0, \infty)} \Phi_n^{*2} d\mu.$$

Passing to the limit we can conclude that

$$\lim_{n\to\infty} \int_{[0, \infty)} \Phi_n^2 d\mu = \int_{[0, \infty)} \Phi^2 d\mu.$$

This was to be proven. ■

Theorem 6.6.3 (Agbeko, [9]) *The subset $\mathfrak{A}$ is a dense set in $\mathcal{Y}_{\text{conc}}$.*

Proof. Let $\Phi \in \mathcal{Y}_{\text{conc}}$ be arbitrary. For every index $n \geq 1$, define $\Phi_n^* := (\Phi_n(1))^{-1}\Phi_n$, where $\Phi_n = \Phi^{n/(n+1)}$. We need to prove that

$$\lim_{n\to\infty} d(\Phi,\, \Phi_n) = \lim_{n\to\infty} \int_{[0,\,\infty)} (\Phi - \Phi_n)^2\, d\mu = 0.$$

In fact, fix arbitrarily an index $n \geq 1$. Then

$$\int_{[0,\,\infty)} (\Phi - \Phi_n)^2\, d\mu = \int_{[0,\,\infty)} \Phi_n^2 d\mu + \int_{[0,\,\infty)} \Phi^2 d\mu - 2\int_{[0,\,\infty)} \Phi\Phi_n d\mu. \tag{42}$$

Then Lemma 6.4.3/4 entails that

$$(\Phi(1))^{-(2n+1)/(n+1)}\, \Phi\Phi_n \leq Z^{(2n+1)/(n+1)} \leq Z^2,$$

since function $Z(x) \geq 1$ for all $x \in [0,\,\infty)$ and sequence $\left(\dfrac{2n+1}{n+1}\right)$ tends increasingly to 2. On the other hand

$$\lim_{n\to\infty} (\Phi(1))^{-(2n+1)/(n+1)}\, \Phi(x)\, \Phi_n(x) = (\Phi(1))^{-2}\, \Phi^2(x)$$

for all $x \in [0,\,\infty)$. Then by means of The Dominated Convergence Theorem it follows that

$$\begin{aligned}\lim_{n\to\infty} \int_{[0,\,\infty)} \Phi\Phi_n d\mu &= \lim_{n\to\infty} (\Phi(1))^{(2n+1)/(n+1)} \int_{[0,\,\infty)} (\Phi(1))^{-\frac{2n+1}{n+1}}\, \Phi^{\frac{2n+1}{n+1}} d\mu \\ &= \int_{[0,\,\infty)} \Phi^2 d\mu.\end{aligned} \tag{43}$$

Finally we note that

$$\lim_{n\to\infty} \int_{[0,\,\infty)} \Phi_n^2 d\mu = \int_{[0,\,\infty)} \Phi^2 d\mu, \tag{44}$$

by Theorem 6.6.2. Therefore, combining the results (42)–(44), we get $\lim_{n\to\infty} d(\Phi,\, \Phi_n) = 0$. We can thus conclude on the validity of the theorem. ■

CHAPTER VII

SOME COMPUTATIONAL ASPECTS

Optimal measure theory can be applied in many fields such as genetic algorithms, neural network, computer algebra, artificial intelligence and it can be used to substitute the Sugeno integral.

7.1 Algorithmic determination of optimal measure from data

In fuzzy sets theory the crux was how to determine the values of the fuzzy measure in a given real problem. To achieve that goal the Sugeno integral was used alongside the so-called genetic algorithm to solve it (see [66]), say. The Sugeno integral with respect to a given fuzzy measure μ is regarded as a multi-input single-output system. The input is the integrand, i.e. the vector $(f(\omega_1), \ldots, f(\omega_n))$, while the output is the value of its Sugeno integral $E := (\mathrm{S})\int f d\mu = \sup\{\alpha \wedge \mu(F_\alpha) : \alpha \in [0, 1]\}$, where f is a measurable function defined on a finite measurable space $(\Omega, \mathcal{F})$ and $F_\alpha := \{\omega \in \Omega : f(\omega) \geq \alpha\}$. By repeatedly observing the system $(f(\omega_1), \ldots, f(\omega_n))$ results the following

$f_{11}(\omega_1)$	$f_{12}(\omega_2)$	$\ldots$	$f_{1k}(\omega_n)$	E_1
$f_{21}(\omega_1)$	$f_{22}(\omega_2)$	$\ldots$	$f_{2k}(\omega_n)$	E_2
$\vdots$	$\vdots$		$\vdots$	$\vdots$
$f_{k1}(\omega_1)$	$f_{k2}(\omega_2)$	$\ldots$	$f_{kk}(\omega_n)$	E_k

and we look for an approximate fuzzy measure μ with $E_i = (\mathrm{S})\int f_i d\mu$, $(i = 1, \ldots, k)$, such that the expression

$$e := \sqrt{\frac{1}{k}\sum\nolimits_{i=1}^{k}\left(E_i - (S)\int f_i d\mu\right)^2}$$

is minimized. For more about the genetic algorithm see [41], for example.

An analogical crucial question also arises to know how to determine the range of the optimal measure in a real problem. We shall first formulate some useful problems.

Problem 1 *Let $(\Omega, \mathcal{F})$ be the measurable space with $\Omega = \{1, \ldots, n\}$ and $\mathcal{F} = 2^\Omega$, i.e. $\mathcal{F}$ is the power set of Ω. Write $B_1 := \{1\}, \ldots, B_n := \{n\}$ and let f be a random variable assuming the theoretical values in $[0, \infty)$. Observe k times this measurable function with* *results* $f_1, \ldots, f_k$, *i.e.*

B_1	B_2	$\ldots$	B_n	
$f_1(1)$	$f_1(2)$	$\ldots$	$f_1(n)$	Q_1
$f_2(1)$	$f_2(2)$	$\ldots$	$f_2(n)$	Q_2
$\vdots$	$\vdots$		$\vdots$	$\vdots$
$f_k(1)$	$f_k(2)$	$\ldots$	$f_k(n)$	Q_k

where $Q_i = \dfrac{1}{n}\sum_{j=1}^{n} f_{ij}$ *with* $f_{ij} := f_i(j)$, $j = 1, \ldots, n$, *and* $i = 1, \ldots, k$. *The question is to know which one of these sample averages can "best" approximate the theoretical mathematical expectation.*

To solve Problem 1 we propose to look for an approximation of the theoretical optimal measure p for which $\int_\Omega f_i dp \approx Q_i$, $(i = 1, \ldots, k)$, such that the expression

$$err := \sqrt{\sum_{i=1}^{k} \varepsilon_i^2} = \sqrt{\sum_{i=1}^{k} \left(Q_i - \int_\Omega f_i dp\right)^2}$$

is minimized. Write p_* for the optimal measure p for which the least square is minimal. Now, it is not difficult to see that $\bigvee_{i=1}^{k} \left|Q_i - \int_\Omega f_i dp_*\right| < err$. Let i_0 be the index where the maximum is attained, i.e.

$$\left|Q_{i_0} - \int_\Omega f_{i_0} dp_*\right| = \bigvee_{i=1}^{k} \left|Q_i - \int_\Omega f_i dp_*\right|.$$

Then we can conclude that with respect to the optimal measure p_* the i_0th sample provides us with the best possible sample average.

As we know statistical spaces are not restricted in general to the real line nor to the real vector spaces. For this reason we need to formulate the following problem. We shall then indicate how to reduce Problem 2 to Problem 1.

Problem 2 *Let* $(X, \mathcal{S})$ *be measurable space with* $\mathcal{S}$ *being an arbitrary* σ*-algebra. Fix a partition* $D_1, \ldots, D_n$ *of* X *and consider a random variable* $h : X \to [0, \infty)$, *assuming theoretical values. Observe* k *times this measurable function with the following results*:

D_1	D_2	$\ldots$	D_n	
h_{11}	h_{12}	$\ldots$	h_{1n}	Q_1
h_{21}	h_{22}	$\ldots$	h_{2n}	Q_2
$\vdots$	$\vdots$		$\vdots$	$\vdots$
h_{k1}	h_{k2}	$\ldots$	h_{kn}	Q_k

where h_{ij} *is the observed value of* h *in the* i*th trial on event* D_j, $i = 1, \ldots, k$; $j = 1, \ldots, n$, *and* $Q_i = \frac{1}{n}\sum_{j=1}^{n} h_{ij}$, $i = 1, \ldots, k$.

The question is to know which one of these sample averages can "best" approximate the theoretical mathematical expectation of h.

To solve Problem 2, first write $\mathcal{S}_0 := \sigma(D_1, \ldots, D_n)$. We note that $\mathcal{S}_0$ is a finite σ-algebra and the random variable h is also $\mathcal{S}_0$-measurable. Clearly, $\mathcal{S}_0$ and 2^{Ω} are equinumerous, where $\Omega = \{1, \ldots, n\}$. Then Problem 2 can be reduced to Problem 1 if we define $f_{ij} := h_{ij}$, $i = 1, \ldots, k$; $j = 1, \ldots, n$.

Step 0.

Input: n positive integer
$\Omega = \{1, \ldots, n\}$
$k \times n$ matrix $F = [f(i,j)]_{i,j=1}^{n,k}$
n-dimensional vector Q
error bound ε
$B_j = \{j\}$, $j = 1...n$
$X =$ the power set of Ω.
Generate the set σ of all permutations of $\{1, ..., n\}$.

Step 1.

Generate a decreasing sequence $\alpha(j) \in (0, 1]$, with $\alpha(1) = 1$.

Step 2.

For any permutation $\{n_1, ..., n_n\} \in \sigma$
Put $p(B_j) = \alpha(n_j)$, for $j = 1, ..., n$
Compute the optimal average: $A(i) = \max\{f(i,j) * p(B_j) : j = 1...n\}$, for $i = 1..k$
Compute the corresponding error: $err = \sqrt{\left(\sum_{j=1}^{n} (Q(i) - A(i))\right)^2}$

Step 3.

If $err < \varepsilon$ for some permutation *do*
Find the index i_0 such that $|Q(i_0) - A(i_0)| = \max\{|Q(i) - A(i)| : i = 1...k\}$
Determine $p(B) = \max\{\alpha(n_j) : j \in B\}$, for each $B \in X$
Else GOTO **Step 1**

Step 4.

The outputs
1.) **Best sample:** $f(i_0, 1), \ldots, f(i_0, n)$
2.) **The approximated optimal measure:**

2^Ω	$p(B)$
$\{\}$	0
B_1	$p(B_1)$
$\vdots$	$\vdots$
B_i	$p(B_i)$
$\vdots$	$\vdots$

7.2 A Maple codes solution of Problem 1

```
> with(combinat):
with(stats):
with(VectorCalculus):
st:=time():
bestsample:= proc(k,n,epsz)
local i,j,vel,per,lepes,err1;
global Omega,A,B,X,S,F,alpha,Q,err,p,measure,i_0,setmeasure;
Omega:={};
setmeasure:=array(1..2^n,1..2);
for i from 1 to n do
Omega:=Omega union {i};
od;
S := subsets(Omega):
i:=1;
while not S[finished] do
X[i]:=S[nextvalue]();
i:=i+1:
od;
F:=array(1..k,1..n);
for i from 1 to k do
for j from 1 to n do
F[i,j]:=abs(stats[random, normald](1));
od:
od;
print(`Matrix F:`); print(F);
for j from 1 to n do
B[j]:=j;
od;
Q:=array[1..k];#vector[k];
for i from 1 to k do
Q[i]:=0;
for j from 1 to n do
Q[i]:=Q[i]+F[i,j]/n;
od;
od;
print(`The meanvalues of the rows of matrix F, i.e.  vector Q:`); print(Q);
err1:=10000000;
while (err1>epsz) do
alpha:=[1,op(sort(RandomTools[Generate](list(float(range=0..1),
n-1)),`>`))];
lepes:=1;
err1:=10000000;
print(`The value of alpha:`); print(alpha);
while (err1>epsz and lepes<=n!)  do
per:=permute(n)[lepes];
for j from 1 to n do
p[j]:=alpha[per[j]];
od;   for i from 1 to k do
for j from 1 to n do
if j=1 then A[i]:=F[i,j]*p[j];
```

```
else if A[i]<F[i,j]*p[j]
then A[i]:=F[i,j]*p[j];
fi;
fi;
od;
od;
err:=0;
for i from 1 to k do
err:=(Q[i]-A[i])^2;
od;
err:=sqrt(err);
lepes:=lepes+1;
if err1>err then err1:=err; fi;
od;
print(`The error obtained from this alpha:`);print(err1);
od;
print(`The permutation which provides the optimal measure:`); print(per);
i_0:=1;
for i from 2 to k do
if abs(Q[i]-A[i])>abs(Q[i_0]-A[i_0]) then i_0:=i; fi;
od;
print(`The value of i_0:`);print(i_0);
for i from 1 to 2^n do
measure:=0;
for j from 1 to n do
if j in X[i] then measure:=max(measure,p[j]); fi;
od;
setmeasure[i,1]:=X[i];
setmeasure[i,2]:=
mertek;
od;
print(`A halmazokra kapott measure:`);print(setmeasure);
print(`The run time:`); time()-st;
end;
> bestsample(3,4,0.02);
```

Matrix F:

$$\begin{bmatrix} 1.315575422 & 0.4312628907 & 0.3691538117 & 1.987882081 \\ 0.3806605310 & 1.213901996 & 2.020635570 & 1.033761787 \\ 0.8287258064 & 1.058650159 & 0.3007036528 & 0.6182133403 \end{bmatrix}$$

The mean values of the rows of matrix F, i.e. vector Q:

$TABLE\,([1 = 1.025968551, 2 = 1.162239971, 3 = 0.7015732397])$

The elements of vector α:

$[1, 0.8632200803, 0.000009418418783, 0.0000000009175679917]$

The error obtained from this α:

0.01379951740

The permutation which provides the optimal measure:

$[2, 3, 1, 4]$

The value of i_0:

2

The measure of the sets:

$\{\}$	0
$\{1\}$	0.8632200803
$\{2\}$	0.000009418418783
$\{3\}$	1
$\{4\}$	0.0000000009175679917
$\{1, 2\}$	0.8632200803
$\{1, 3\}$	1
$\{1, 4\}$	0.8632200803
$\{2, 3\}$	1
$\{2, 4\}$	0.000009418418783
$\{3, 4\}$	1
$\{1, 2, 3\}$	1
$\{1, 2, 4\}$	0.8632200803
$\{1, 3, 4\}$	1
$\{2, 3, 4\}$	1
$\{1, 2, 3, 4\}$	1

The run time:

1.812

7.3 *Algorithm for finding the degree of contraction and the positive fixed point*

Let be given a concave Young function Φ and a positive number cc in the neighborhood of 0.

Step 1. Input $\Phi(x)$, cc

Step 2. Compute the derivative $\varphi(x)$ of $\Phi(x)$.

Step 3. Starting from cc find an approximation root for equation $\varphi(x) - 1 = 0$ and put the result into c.

Step 4. *If* $c = 0$ *then* STOP.
else GOTO Step 5.

Step 5. Starting from c apply the FixedPoint algorithm, i.e.
$x_0 := c;\ x_{k+1} := \Phi(x_k);\ k = k + 1.$

7.4 *A Maple program for computing the degrees of contraction and the positive fixed point*

```
> restart;
> t:=array(1..3,1..4): t[1,1]:='k': t[1,2]:='c': t[1,3]:='ido':
t[1,4]:='fixpont':
```

Input section

```
> for k from 2 to 3 do
Phi[k-1]:=x-> (k^(k+1)*x)^(1/(k+2))+(k-1)*(log(x+k)-log(k));
od:
```

The computation of the derivatives

```
> for k from 2 to 3 do
phi[k-1] :=diff(Phi[k-1](x),x);
psi[k-1] :=D(Phi[k-1]);
od:
```

The computation of the h(x):=phi(x)-1 and the function indexes

```
> for k from 2 to 3 do
h[k-1] :=phi[k-1]-1;
t[k,1]:=k-1;
od:
```

The degree of contraction can be obtained as a possible approximated root of function h(x):=phi(x)-1. To do this the bisection method (intervallum-felező eljárás, in Hungarian) is rather preferable to Newton and other methods, say. Because differentiating twice the above two functions can be fatal in terms of time period. The subroutine for these approximations is as follows:

```
> bisect :=proc(fun,x0,x1,max,delta)
local k1,halff,p0,p1,p2,a,b;
k1 := 0:    a:=x0;b:=x1:
#while er>delta do
halff:=(a+b)/2:
p0 := evalf(subs(x=a,fun)):
p1 := evalf(subs(x=b,fun)):
p2 := evalf(subs(x=halff,fun)):
```

```
for k1 from 1 to max do
if (evalf(p0*p2)<0) then
b:=halff:
else
a:=halff:
end if:
halff:= (a+b)/2:
p0 :=evalf(subs(x=a,fun)):
p1 := evalf(subs(x=b,fun)):
p2 := evalf(subs(x=halff,fun)):
end do:
RETURN(halff):
end proc:
>  for k from 2 to 3 do
w[k-1]:=plot({Phi[k-1](x)},x=0..30,color=black):
PP[k-1]:=evalf(subs(x=20+1.5,Phi[k-1](x))):
od:
wx:=plot(x,x=0..30,color=red):
>  with(plots):
for k from 2 to 3 do
u[k-1]:=textplot([20,PP[k-1],`Phi(x)`],align={BELOW,RIGHT}):
od:
ux:=textplot([20,20,`y =x`],align={BELOW,RIGHT}):
for k from 2 to 3 do
display({wx,w[k-1],ux,u[k-1]});
od;
```

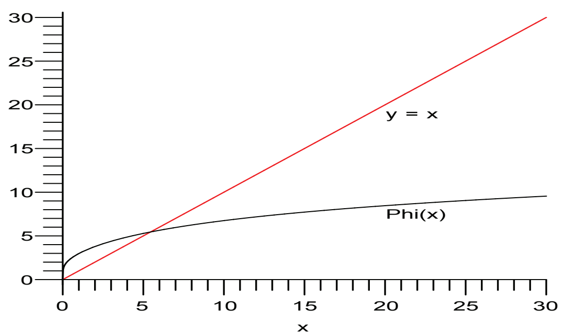

Figure 1: The joint plot of $\Phi_1(x)$ and the line $y = x$.

```
>
>  for k from 2 to 3 do st:= time():
c[k]:=bisect(h[k-1],0.00000002,20,40);
```

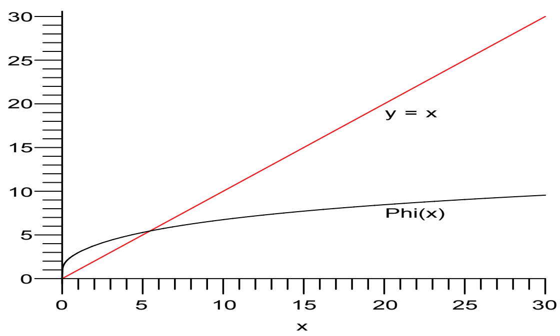

Figure 2: The joint plot of $\Phi_2\,(x)$ and the line $y = x$.

```
ido[k]:=time() - st; t[k,2]:=c[k]; t[k,3]:=ido[k];
od:
```

Algorithm to compute the approximation of the fixed point.

```
>  fixedpoint := proc(x0,max,k,t)
local k1,p0,p1;
k1 := 0;
p0 :=evalf(x0);
printf(" P%g = %g \n",k1,p0);
p1 := p0;
for k1 from 1 to max do
p0 := evalf(p1);
p1 := g(p0);
printf(" P%g = %g \n",k1,p1);
end do;
print(`g(x) = `,g(x));
printf(" P = %g \n",p1);
printf("g(P) = %g \n",g(p1));
t[k,4]:=evalf(p1);  RETURN(p1);
end:
```

Starting with x0 = c we compute the approximation of the fixed point, where phi(c) = 1

```
>  for k from 2 to 3 do
g:= unapply(Phi[k-1](x),x):
fixedpoint(c[k],10,k,t):
od:
```

$\Phi_1(x) = (8x)^{1/4} + \ln(x+2) - \ln 2$	$\Phi_2(x) = (81x)^{1/5} + \ln(x+3)^2 - \ln 9$
P0 = 0.601473	P0 = 0.96497
P1 = 1.744000	P1 = 2.94888
P2 = 2.559690	P2 = 4.35889
P3 = 2.95136	P3 = 5.02734
P4 = 3.11085	P4 = 5.29481
P5 = 3.17175	P5 = 5.39502
P6 = 3.19445	P6 = 5.43167
P7 = 3.20283	P7 = 5.44495
P8 = 3.20592	P8 = 5.44975
P9 = 3.20705	P9 = 5.45148
P10 = 3.20747	P10 = 5.45211

>

In the following table the columns contains respectively the indexes, the solution of phi(c)=1, the time needed to obtain c as well as the fixed point of the functions

```
> print(t);
```

4	*c*	*ido*	*fixpont*
1	0.6014731621	0.016	3.207470869
2	0.9649698547	0.031	5.452106877

>

Comparison of the distance between the images of two points with a constant multiple of their distance

```
> for k from 2 to 3 do
#cc:=0.06:
#print(k-1);
plot3d(psi[k-1](c[k])*abs(x-y), x = c[k] ..  50,
y = c[k] ..50,axes=BOXED):
plot3d(abs(Phi[k-1](x)-Phi[k-1](y)),x = c[k] ..  50,
y = c[k] ..  50,axes=BOXED):
plot3d([abs(Phi[k-1](x)-Phi[k-1](y)), psi[k-1](c[k])*abs(x-y)],
x= c[k] ..  50, y = c[k] ..  50,axes=BOXED):
od;
```

>

Comparison of the differential rate with the appropriate value of the derivative

```
> for k from 2 to 3 do
plot3d(psi[k-1](c[k]), x = c[k]..  50, y = c[k] ..  50,axes=BOXED):
plot3d(abs((Phi[k-1](x)-Phi[k-1](y))/(x-y)), x = c[k] ..  50,
y =c[k] ..  50,axes=BOXED):
plot3d([abs((Phi[k-1](x)-Phi[k-1](y))/(x-y)), psi[k-1](c[k])],
x =c[k] ..  50, y = c[k] ..  50,axes=BOXED):
od;
```

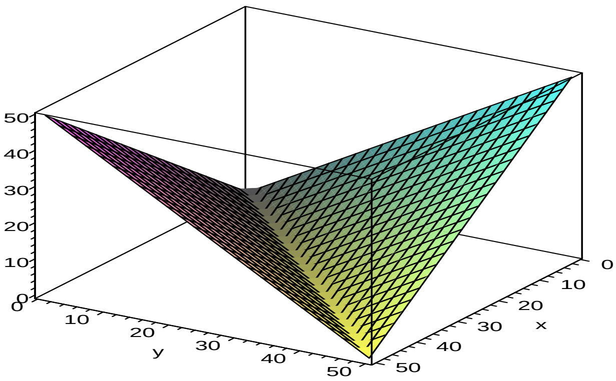

Figure 3: Plot of $c_1 |x - y|$

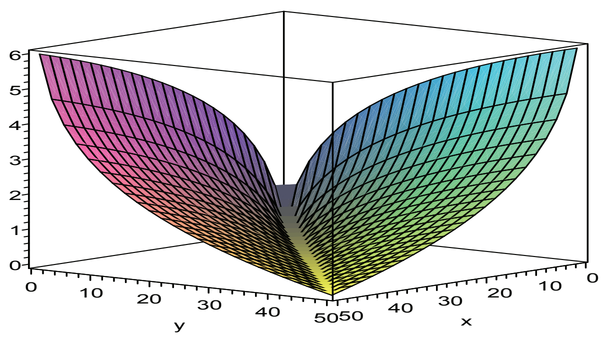

Figure 4: Plot of the distance $|\Phi_1(x) - \Phi_1(y)|$.

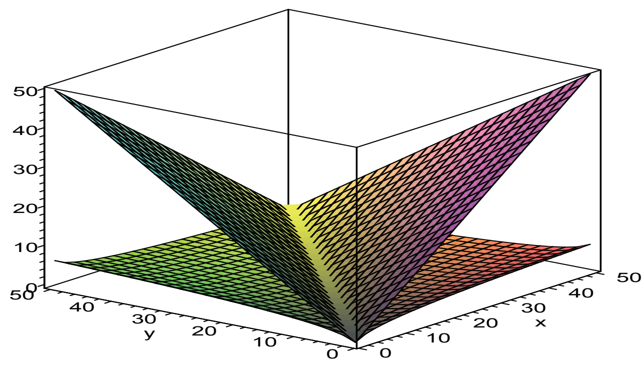

Figure 5: The plot of the above two configurations.

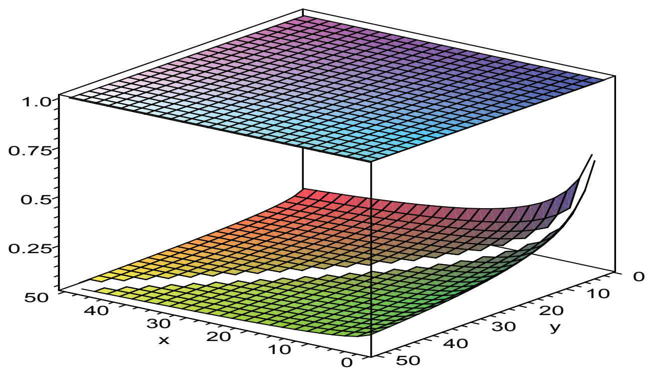

Figure 6: Plot of $\varphi_1(c_1)$ and plot of the ratio $\frac{|\Phi_1(x)-\Phi_1(y)|}{|x-y|}$.

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