Deflections and Slopes of Shafts

Study aid

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Calculation of deflections and slopes of shafts

Modelling a shaft as a simply supported beam or an overhanging beam (both statically determinate), and applying the principle of superposition, we examine the following cases.

The simply supported beam being subject to just a concentrated force, F a distance *a* from left support (pin connection) and *b* from right support (roller connection), shown in Fig. 1.

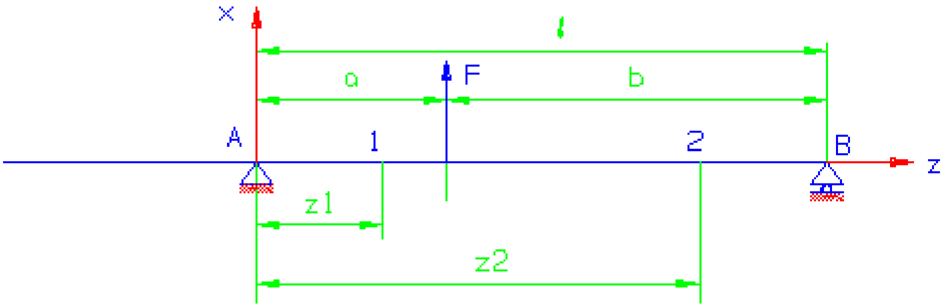

Fig. 1. The simply supported beam is loaded between the supports with concentrated force, placed in a Cartesian coordinate system

The overhanging beam being subject to a concentrated force, F at the free end, shown in Fig. 2.

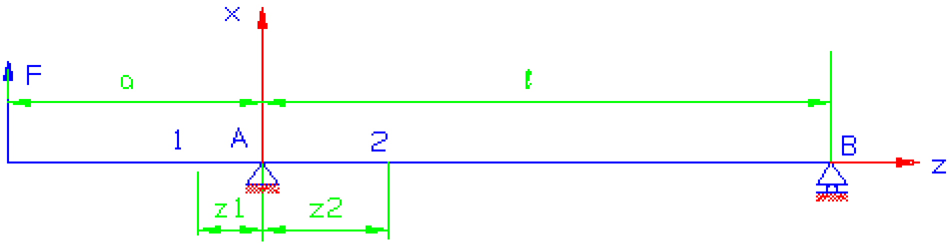

Fig. 2. The overhanging beam is loaded at the free end with concentrated force, placed in a Cartesian coordinate system

In both load cases, at points 1 and 2, the deflections of the beam in the x direction and the slope of the beam can be determined, i.e. $x1$, $x2$, $\vartheta 1$ and $\vartheta 2$, as the function of the z coordinate. From these, the deflection and slope of the beam can be calculated at each selected point, such as x_F, ϑ_F, ϑ_A and ϑ_B. If forces act on the beam both at the free end and between the support, the resulting deflection function will be the sum of the functions obtained from the above two load cases, based on the principle of superposition.
In the following, the elastic curve of the beam is calculated according to Castigliano's theorem. In the case of a statically determined structure and a prismatic beam, it is easier to obtain the result by solving the differential equation of the elastic curve, but Castigliano's theorem can be used much more generally, so the use of this method is presented here.
The work done in bending the beam by external forces, is stored as strain energy, *U*. According to Castigliano's theorem, the partial derivative of the total strain energy of the beam (or system) with respect to the force or pair of forces concerned, is equal to the deflection or slope of the given point of the beam (or system) in the direction of the particular force, *F* or couple, *M*.

$$u_i = \frac{\partial U}{\partial F_i}, \text{ or } \vartheta_i = \frac{\partial U}{\partial M_i}. \qquad (1)$$

At any point of the structure, the projection of the deflection or twist in any direction can be calculated if a force Q (of any size) is applied in the direction of the desired deflection at the

desired location, or a couple Q (of any size) is applied in the direction of the desired twist and is produced together with it the strain energy of the beam (or system), then we get the partial derivative of it with respect to Q, and at the last, by substituting Q=0, we obtain the desired component of the deflection or slope.
Q is called a generalized force (it can be a force or a couple), and q is called a generalized displacement (deflection or twist). If we need to calculate a deflection or slope at a point where no force or couple was originally acting, then with the help of the introduced generalized force and generalized displacement, the Castigliano theorem
will have the form of

$$q = \left.\frac{\partial U}{\partial Q}\right|_{Q=0} \tag{2}$$

[1].
The loads in the beam consist of the original force system (indicated by a zero subscript) and the generalized force Q (and its reaction system) additionally applied to the beam.

$$M_h = M_{h0} + Q \cdot m, \tag{3}$$

where the relative load (here relative bending moment) is

$$m = \frac{M_{hQ}}{Q}. \tag{4}$$

The dimension of m is mm if Q is a force, and dimensionless if Q is a couple.
If the stain energy resulting from only bending is considered as the standard, the strain energy is written in the form of

$$U = \int_{(L)} \frac{(M_{h0} + Q \cdot m)^2}{2\, I \cdot E} ds. \tag{5}$$

Hence the generalized displacement, according to (2) is

$$q = \left.\frac{\partial U}{\partial Q}\right|_{Q=0} = \left.\int_{(L)} \frac{(M_{h0} + Q \cdot m)m}{I \cdot E} ds\right|_{Q=0} = \int_{(L)} \frac{M_{h0} \cdot m}{I \cdot E} ds,$$

that is

$$q = \int_{(L)} \frac{M_{h0} \cdot m}{I \cdot E} ds. \tag{6}$$

In the case of prismatic and homogeneous isotropic bars, both the area moment of inertia and the modulus of elasticity are constant (independent of *s*) so we obtain the form of

$$q = \frac{1}{I\,E} \int_{(L)} M_{h0}\; m\, ds. \tag{7}$$

Using the designations of Fig. 1 and Fig. 2 the deflection of a straight bar can be obtain by the integral of the product of the functions of the original load and the relative loads, that is

$$q = \frac{1}{I\,E} \int_{(L)} M_{h0}(z)\; m(z)\, dz. \tag{8}$$

Deformation of a straight, simply supported beam, loaded by concentrated force between the supports

Let’s consider a simply supported beam according to Figure 1 and plot the functions of bending moments of equation (8). Figure 3 shows the change of the torque associated with the original load along z, and the relative loads associated with the generalized forces Q acting at points 1 and 2. The dimensionless ones belong to the couples, those with the dimension of *m* belong to the force. The directionality of the generalized force was taken everywhere in the positive direction of the x-coordinate, and in the positive direction of rotation of the right-handed coordinate system, and accordingly we plot the loading diagram of the beam with the original load, and the diagrams of the relative loads.

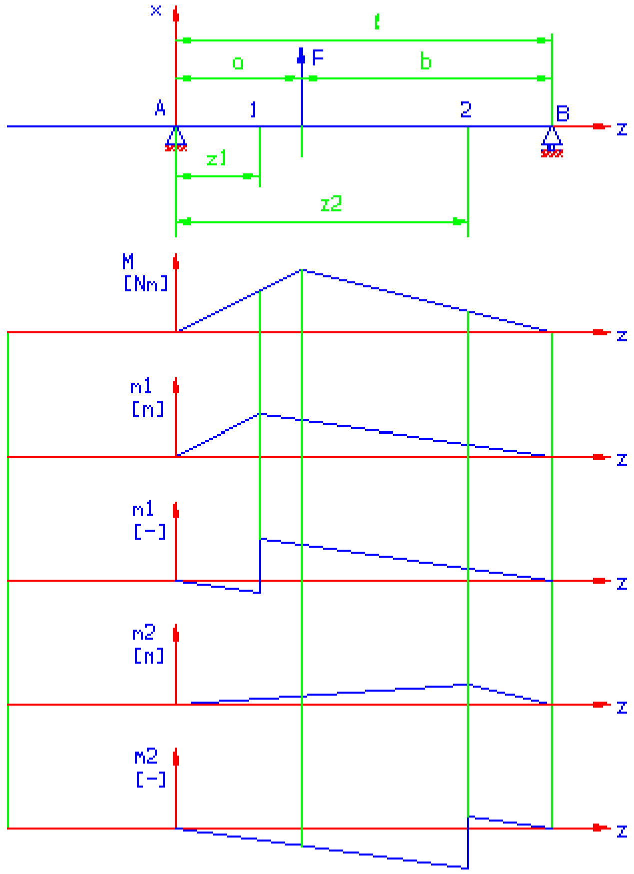

Fig. 3. The bending moment of the actual force system and the relative loads at a simply supported beam (loaded between the supports)

The system of reactions can be calculated with the static equations, the bending moment and relative moment functions are equations of lines. They are straight lines, as the loads are concentrated forces or concentrated couples. The equation of a line, given its two points is

$$x - x_0 = \frac{x_1 - x_0}{z_1 - z_0}\left(z - z_0\right), \tag{9}$$

where *x* can be substituted by the function of bending moment or relative moments, and the two given points are P_0 and P_1 .

Table 1. summarize the reactions at A pin and at B roller, and the equations of bending moments at stage 1 and 2.

The functions of deflection can be computed by equation (8) and the Table 1. The functions of moments are continuous only intermittently, so the integration can be made only domain by domain. The domains can be assigned by figure 3. There are different functions for the deflection and slope at the stage 1 and 2 (to the left or right of the force).

Table 1. Functions of moments of the simply supported beam

Forces or couples, and loading diagrams		Reactions		The bending moment and the relative moments maximum value and functions of z		
		A	B	maximum	Stage 1.	Stage 2.
+F	$M(z)$ [Nm]	$-F\frac{b}{\ell}$	$-F\frac{a}{\ell}$	$F\frac{ab}{\ell}$	$F\frac{b}{\ell}z$	$-F\frac{a}{\ell}(z-\ell)$
+Q	$m_1(z)$ [m]	$-\frac{\ell - z_1}{\ell}$	$-\frac{z_1}{\ell}$	$\frac{\ell - z_1}{\ell}z_1$	$\frac{\ell - z_1}{\ell}z$	$-\frac{z_1}{\ell}(z-\ell)$
+Q	$m_1(z)$ [–]	$-\frac{1}{\ell}$	$\frac{1}{\ell}$	1	$\frac{1}{\ell}z$	$\frac{1}{\ell}(z-\ell)$
+Q	$m_2(z)$ [m]	$-\frac{\ell - z_2}{\ell}$	$-\frac{z_2}{\ell}$	$\frac{\ell - z_2}{\ell}z_2$	$\frac{\ell - z_2}{\ell}z$	$-\frac{z_2}{\ell}(z-\ell)$
+Q	$m_2(z)$ [–]	$\frac{1}{\ell}$	$-\frac{1}{\ell}$	1	$-\frac{1}{\ell}z$	$-\frac{1}{\ell}(z-\ell)$

To the left of the force F

$$q = \frac{1}{I\,E}\int_{(L)} M_{h0}(z)\,m_1(z)\,dz$$

$$I\,E\,q = \int_0^{z1} M_1(z)\,m_1(z)\,dz + \int_{z1}^{a} M_1(z)\,m_2(z)\,dz + \int_a^{\ell} M_2(z)\,m_2(z)\,dz \qquad (10)$$

The generalized displacement of formula (10) specifies both the deflection of the beam in direction of x, f_x and the slope around the axis y, ϑ_y, substituting the moments of first row and the relative moments of the second (for deflection) or third rows (for slope) of Table 1. The obtained equations from the first and second rows are

$$I\,E\,f_{x1} = \int_0^{z1} F\frac{b}{\ell}z\cdot\frac{\ell - z_1}{\ell}z\,dz + \int_{z1}^{a} F\frac{b}{\ell}z\cdot\left(-\frac{z_1}{\ell}(z-\ell)\right)dz + \int_a^{\ell}\left(-F\frac{a}{\ell}(z-\ell)\right)\cdot\left(-\frac{z_1}{\ell}(z-\ell)\right)dz$$

$$\frac{6\,I\,E\,\ell}{F}f_{x1} = 6b\left(1-\frac{z_1}{\ell}\right)\int_0^{z1} z^2\,dz + 6b\frac{z_1}{\ell}\int_{z1}^{a}\ell z - z^2\,dz + 6a\frac{z_1}{\ell}\int_a^{\ell}(z-\ell)^2\,dz$$

$$\frac{6\,I\,E\,\ell}{F}f_{x1} = 2b\left(1-\frac{z_1}{\ell}\right)\left[z^3\right]_0^{z1} + 3bz_1\left[z^2\right]_{z1}^{a} - 2\frac{bz_1}{\ell}\left[z^3\right]_{z1}^{a} + 2a\frac{z_1}{\ell}\left[(z-\ell)^3\right]_a^{\ell}$$

$$\frac{6\,I\,E\,\ell}{F}f_{x1} = 2b\left(1-\frac{z_1}{\ell}\right)z_1^3 + 3bz_1\left(a^2 - z_1^2\right) - 2\frac{bz_1}{\ell}\left(a^3 - z_1^3\right) + 2a\frac{z_1}{\ell}\left[(\ell-\ell)^3 - (a-\ell)^3\right]$$

$$\frac{6\,I\,E\,\ell}{F}f_{x1} = +3bz_1a^2 - bz_1^3 - 2\frac{ba^3z_1}{\ell} - 2a\frac{z_1}{\ell}(a-\ell)^3$$

$$\frac{6\,I\,E\,\ell}{F}f_{x1} = -bz_1^3 + 3bz_1a^2 - 2\frac{ba^3z_1}{\ell} - 2a\frac{z_1}{\ell}(-b)^3$$

$$\frac{6\,I\,E\,\ell}{F}f_{x1} = -bz_1^3 + \left(3ba^2 - 2ab\frac{a^2-b^2}{\ell}\right)z_1$$

$$\frac{6\,I\,E\,\ell}{F} f_{x1} = -bz_1^3 + \left(3ba^2 - 2ba^2 + 2ab^2\right)z_1$$

$$\boxed{\frac{6\,I\,E\,\ell}{F} f_{x1} = -bz_1^3 + ab(a+2b)z_1}, \qquad (11)$$

where the deflection of f_{x1} is computable from. The other deflections are available similarly. The slope around the axis y, also to the left of the force can be obtained by the equation (10), using the first and third rows of Table 1., i.e.

$$I\,E\,\vartheta_{y1} = \int_0^{z1} F\frac{b}{\ell} z \cdot \left(-\frac{1}{\ell} z\right) dz + \int_{z1}^{a} F\frac{b}{\ell} z \cdot \left(-\frac{1}{\ell}(z-\ell)\right) dz + \int_a^{\ell} \left(-F\frac{a}{\ell}(z-\ell)\right) \cdot \left(-\frac{1}{\ell}(z-\ell)\right) dz$$

$$\frac{I\,E\,\ell}{F}\vartheta_{y1} = -\frac{b}{\ell}\int_0^{z1} z^2 dz - \frac{b}{\ell}\int_{z1}^{a} z^2 - z\cdot\ell dz + \frac{a}{\ell}\int_a^{\ell}(z-\ell)^2\ dz$$

$$\frac{I\,E\,\ell}{F}\vartheta_{y1} = -\frac{b}{\ell}\int_0^{a} z^2 dz + b\int_{z1}^{a} z\ dz + \frac{a}{\ell}\int_a^{\ell}(z-\ell)^2\ dz$$

$$\frac{6\,I\,E\,\ell}{F}\vartheta_{y1} = -2\frac{ba^3}{\ell} + 3b\left(a^2 - z_1^2\right) - 2\frac{a}{\ell}(a-\ell)^3$$

$$\frac{6\,I\,E\,\ell}{F}\vartheta_{y1} = -2\frac{ba^3}{\ell} + 3ba^2 - 3bz_1^2 + 2\frac{a}{\ell}b^3$$

$$\frac{6\,I\,E\,\ell}{F}\vartheta_{y1} = -2ab\frac{a^2-b^2}{\ell} + 3ba^2 - 3bz_1^2$$

$$\frac{6\,I\,E\,\ell}{F}\vartheta_{y1} = -2ab(a-b) + 3aba - 3bz_1^2$$

$$\frac{6\,I\,E\,\ell}{F}\vartheta_{y1} = +a^2b - 2ab^2 - 3bz_1^2$$

$$\boxed{\frac{6\,I\,E\,\ell}{F}\vartheta_{y1} = -3bz_1^2 + ab(\ell+b)} \qquad (12)$$

where the slope of ϑ_{y1} is computable from. The other slopes are available similarly.

To the right of the force F

$$I\,E\,q = \int_0^{a} M_1(z)\,m_1(z)\,dz + \int_a^{z2} M_2(z)\,m_1(z)\,dz + \int_{z2}^{\ell} M_2(z)\,m_2(z)\,dz \qquad (13)$$

equation determines the deflection and slope of beam, substituting the relative moments from fourth and fifth rows of Table 1. The obtained equations are

$$\boxed{\frac{6\,I\,E\,\ell}{F} f_{x2} = -a(\ell - z_2)^3 + ab(2a+b)(\ell - z_2)} \qquad (14)$$

$$\boxed{\frac{6\,I\,E\,\ell}{F}\vartheta_{y2} = 3a(\ell - z_2)^2 - ab(a+2b)} \qquad (15)$$

Deflection and slope of an overhanging beam loaded at the free end by concentrated force F

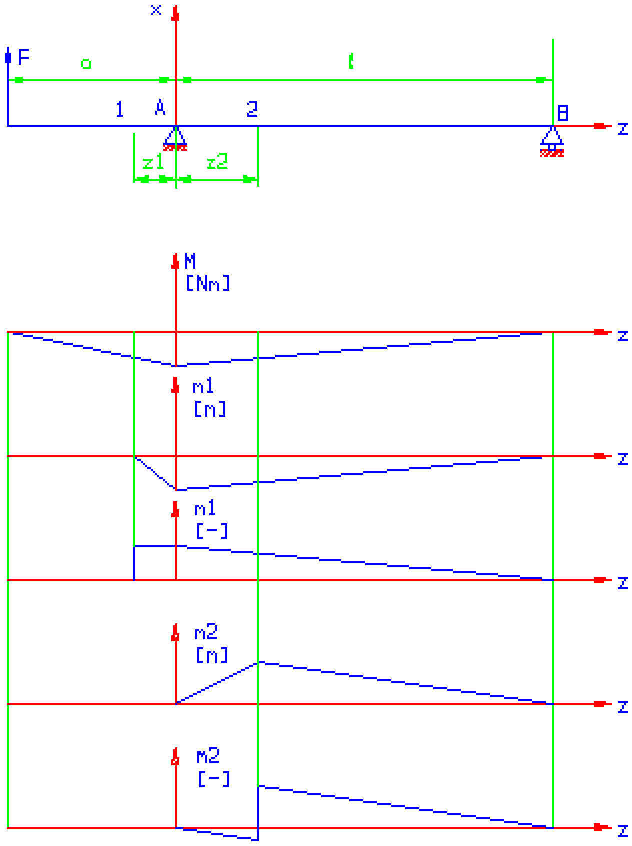

Fig. 4. The bending moment of the actual force system and the relative loads at an overhanging beam, loaded at the free end

The functions of deflection at the overhanging beam, loaded at the free end, can be computed by equation (8) and the Table 2. The functions of moments are continuous only intermittently, too, so the integration can be made only domain by domain. The domains can be assigned by figure 4. There are different functions for the deflection and slope to the left or right of the left support *A*.

To the left of the support A

$$I\,E\,q = \int_{-z1}^{0} M_1(z)\, m_1(z)\, dz + \int_{0}^{\ell} M_2(z)\, m_2(z)\, dz\,. \tag{16}$$

The generalized displacement of formula (16) specifies both the deflection of the beam in direction of *x*, f_x and the slope around the axis *y* , ϑ_y, substituting the moments of first row and the relative moments of the second (for deflection) or third rows (for slope) of Table 2 . The obtained equations from the first and second rows are

$$\boxed{\frac{6\,I\,E\,\ell}{F} f_{x1} = \ell\left(-z_1^3 + 3az_1^2 + 2a\ell z_1\right)} \tag{17}$$

$$\boxed{\frac{6\,I\,E\,\ell}{F} \vartheta_{y1} = -\ell\left(-3z_1^2 + 6az_1 + 2a\ell\right)} \tag{18}$$

Table 2. Functions of moments of the overhanging beam, loaded at the free end

Forces or couples, and loading diagrams		Reactions		The bending moment and the relative moments maximum value and functions of z		
		A	B	maximum	Stage 1.	Stage 2.
+F	$M(z)$ [Nm]	$-F\left(1+\frac{a}{\ell}\right)$	$F\frac{a}{\ell}$	$-Fa$	$-F(z+a)$	$F\frac{a}{\ell}(z-\ell)$
+Q	$m_1(z)$ $[m]$	$-\left(1+\frac{z_1}{\ell}\right)$	$\frac{z_1}{\ell}$	$-z_1$	$-(z+z_1)$	$\frac{z_1}{\ell}(z-\ell)$
+Q	$m_1(z)$ $[-]$	$\frac{1}{\ell}$	$-\frac{1}{\ell}$	1	1	$-\frac{1}{\ell}(z-\ell)$
+Q	$m_2(z)$ $[m]$	$-\frac{\ell-z_2}{\ell}$	$-\frac{z_2}{\ell}$	$\frac{\ell-z_2}{\ell}z_2$	$\frac{\ell-z_2}{\ell}z$	$-\frac{z_2}{\ell}(z-\ell)$
+Q	$m_2(z)$ $[-]$	$\frac{1}{\ell}$	$-\frac{1}{\ell}$	1	$-\frac{1}{\ell}z$	$-\frac{1}{\ell}(z-\ell)$

To the right of the support A

$$I\,E\,q = \int_0^{z2} M_2(z)\,m_1(z)\,dz + \int_{z2}^{\ell} M_2(z)\,m_2(z)\,dz \qquad (19)$$

equation determines the deflection and slope of beam, substituting the relative moments from fourth and fifth rows of Table 2. The obtained equations are

$$\frac{6\,I\,E\,\ell}{F} f_{x2} = -a\left(z_2^3 - 3\ell z_2^2 + 2\ell^2 z_2\right) \qquad (20)$$

$$\frac{6\,I\,E\,\ell}{F} \vartheta_{y2} = a\left(-3z_2^2 + 6\ell z_2 - 2\ell^2\right) \qquad (21)$$

The deflection of the shaft in the direction of axis x is equal to the sum of the two functions of deflection $f_x(z)$, considering the sign of them, of course. If there are forces at the plane yz, too, then there should be summarized two other deflection functions $f_y(z)$considering their signs, and at last the full deflection at a given coordinate of z (for example at the location of a meshing gear z_F) can be computed as

$$f = \sqrt{f_x^2(z_F) + f_y^2(z_F)} \qquad (22)$$

The slope of a given point of axis of the shaft can be computed similarly by the equation

$$\vartheta = \sqrt{\vartheta_x^2(z_F) + \vartheta_y^2(z_F)}\,. \qquad (23)$$

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