2. Waves in Elastic Media, Mechanical Waves

Wave motion appears in almost every branch of physics. We confine our attention to waves in deformable or elastic media. These waves, for example ordinary sound waves in air, are called mechanical waves. If a vibratory disturbance occurs at any point in an elastic medium, this disturbance will be transmitted from one layer to the next through the medium, because of the elastic forces on adjacent layers. The medium itself does not move as a whole.

In a mechanical wave the energy is also transmitted form one point to the next, by the motion or propagation of a disturbance, without any corresponding bulk motion of the matter itself.

A material medium is necessary for the transmission of mechanical waves. Such a medium is not needed to transmit electromagnetic waves. It propagates in vacuum.

2.1 Types of Waves

We can distinguish different kinds of mechanical waves by considering how the motions of the particles of matter are related to the direction of propagation of the waves themselves. The waves are called longitudinal if the motion of each individual particle of the medium is in the same direction as the wave propagation through that point. Sound waves in air are longitudinal waves.
If the motions of the matter paticles are perpendicular to the direction of propagation of the wave itself, we then have a transverse wave. If we have a stretched string and we oscillate the endpoint perpendicular to the direction of the string, the wave is transverse wave.

If our motion is periodic, we produce a periodic train of waves in which each particle of the string has a periodic motion. The simplest special case of a periodic wave is a simple harmonic wave, which gives each particle a simple harmonic motion.

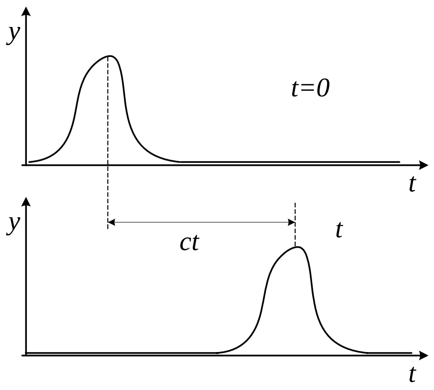
A wave front is a surface connecting medium particles all of which are moving in the same manner at any given moment. A wave called spherical wave or plane wave, if the wave front has the corresponding shape.

2.2 Traveling waves

Let us consider a long string stretched in the x-direction along which a transverse wave is traveling. At $t = 0$ the shape of the string can be represented by $y = f(x)$, where y is the transverse displacement of the string at the position x. Experiment shows that as time goes on such a wave travels along the string without changing its form, if the internal frictional losses are small enough. At some time t later the wave has travelled a distance ct to the positive x-direction, where c is the magnitude of the wave velocity.

The equation of the curve at the time t:

$$y = f\left(x - ct\right)$$



The above equation is the mathematical form of a pulse traveling along the positive *x*-direction.

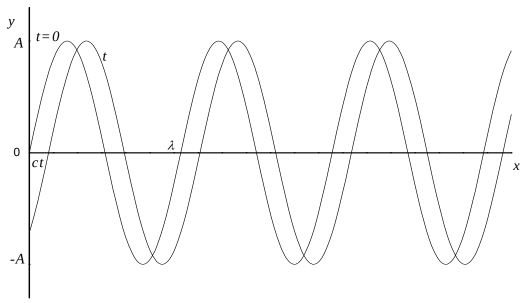
In more general, this *y(x, t)* function may represent several different physical quantities, such as the deformation in a solid, the pressure in a gas, and so on.

An especially interesting case is that in which *y(x, t)* a sinusoidal or harmonic function. Suppose a wave travels from left to right, the direction of increasing *x*. In a sinusoidal wave, all points in the medium move with the same frequency but with phase difference. Suppose, that at time $t = 0$, we have a wave train along the string given by:

$$y = A\sin\frac{2\pi}{\lambda}x$$

The wave shape is a sine curve, the maximum displacement is *A*. The value of the transverse displacement is the same at x, $x+\lambda$, $x+2\lambda$..., and so on. The symbol λ is called wavelength of the wave train and represents the distance between two adjacent points in the wave having the same phase. As the time goes on the wave travels to the right with a phase speed *c*. Hence the equation of the wave at time *t* is:

$$y(x,t) = A\sin\frac{2\pi}{\lambda}(x - ct)$$



The period *T* is the time required for the wave to travel a distance of one wavelength λ, so that

$$\lambda = cT\ .$$

Putting this relation to the above equation:

$$y(x,t) = A\sin\frac{2\pi}{cT}(x - ct) = A\sin\frac{2\pi}{T}\left(\frac{x}{c} - t\right) = A\sin\omega\left(\frac{x}{c} - t\right)$$

Where the angular frequency ω is defined:

$$\omega = \frac{2\pi}{T},$$

so the equation of a sine wave traveling to the right:

$$y(x,t) = A\sin\omega\left(\frac{x}{c} - t\right)$$

It is one way to describe a simple harmonic traveling wave. Let us now consider the other way to obtain the mathematical form of a sinusoidal traveling wave.

Suppose the displacement of a particle at he left end of the string (at $x = 0$), where the motion originates, is given by

$$y = -A\sin\omega t$$

The cause of the negative sign is shown on the previous figure. The oscillating point at $x = 0$ position starts to move into the negative *y* direction. The time required for the wave disturbance to travel from $x = 0$, to some point *x* to the right is given by $\frac{x}{c}$, where *c* is the phase speed.

The motion of point *x* at time *t* is the same as the motion of point $x = 0$ at the earlier time $\left(t - \frac{x}{c}\right)$. Thus the displacement of point *x* at time *t* is obtained by replacing *t* in the above equation by $\left(t - \frac{x}{c}\right)$, we find

$$y(x,t) = -A\sin\omega\left(t - \frac{x}{c}\right)$$

The relation $\sin(-\alpha) = -\sin\alpha$ is known, that is

$$y(x,t) = A\sin\omega\left(\frac{x}{c} - t\right).$$

The same formula as we have already.

To reduce the equation to a more compact form we define the wave number *k* as:

$$k = \frac{2\pi}{\lambda}.$$

We have already used $\omega = \dfrac{2\pi}{T}$, that is

$$\frac{\omega}{c} = \frac{2\pi}{cT} = \frac{2\pi}{\lambda} = k$$

$$y(x,t) = A\sin\left(\frac{\omega}{c}x - \omega t\right)$$
$$y(x,t) = A\sin(kx - \omega t).$$

For a sine wave traveling to the left:

$$y(x,t) = A\sin(kx + \omega t).$$

In the traveling waves we have assumed that the displacement *y* is zero at the position $x = 0$. This, of course, need not to be the case. The general expression for a sinusoidal wave traveling in the +*x* direction:

$$y(x,t) = A\sin(kx - \omega t - \varphi)$$

Where the quantity in parentheses is called the phase and φ is called phase constant.

2.3 Superposition and interference

It is expeimental fact, that for many kinds of waves two or more waves can traverse the same space independently of one another. It means that the displacement of any particle at a given time is simply the sum of the displacements that the individual waves alone would give it. This type of addition is called superposition. The wave trains involved are said to interfere.

Interference refers to the physical effects of superimposing two or more wave trains. Let us consider two waves of equal frequency and amplitude traveling with the same speed in the same direction $(+x)$ but with a phase difference φ between them. The equation of the two waves are:

$$y_1 = A\sin(kx - \omega t)$$
$$y_2 = A\sin(kx - \omega t - \varphi).$$

Now let us find the resultant wave, if superposition occurs, that is the sum:

$$y = y_1 + y_2 = A[\sin(kx - \omega t) + \sin(kx - \omega t - \varphi)]$$

From the trigonometric equation for the sum of the sine of two angles:

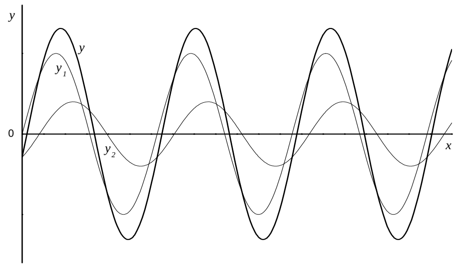
$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}$$

we obtain

$$y = 2A\sin\left(kx - \omega t - \frac{\varphi}{2}\right)\cos\frac{\varphi}{2}, \text{ or}$$

$$y = 2A\cos\frac{\varphi}{2}\sin\left(kx - \omega t - \frac{\varphi}{2}\right).$$

This resultant wave corresponds to a new wave having the same frequency but with am amplitude $2A\cos\frac{\varphi}{2}$.



The very special cases:

When $\varphi = 0$, $\cos\frac{\varphi}{2} = 1$, the two waves have the same phase. The waves interfere constructively, and the resultant amplitude is 2*A*. If $\varphi = 180^o$, $\cos\frac{\varphi}{2} = 0$, the two waves are in opposite phase, they interfere destructively, and the resultant amplitude is zero.

The principle of superposition is of central importance in all types of wave motion. It applies not only to waves on a string, but also to sound waves, electromagnetic waves (such as light) and all other wave phenomena in which the wave equation is linear.

2.4 Standing Waves

Let us tie one end of an elastic string to a support and oscillate on the other end. We can see a wave train traveling towards the support. The traveling wave in the string is reflected from the boundary, which is the support. Such reflection gives rise to a wave traveling in the string in the opposite direction. The reflected wave adds to the incident wave according to the principle of superposition.

Consider now two wave trains of the same frequency, speed and amplitude which are traveling in opposite direction along a string. Two such waves may be represented by the next equations:

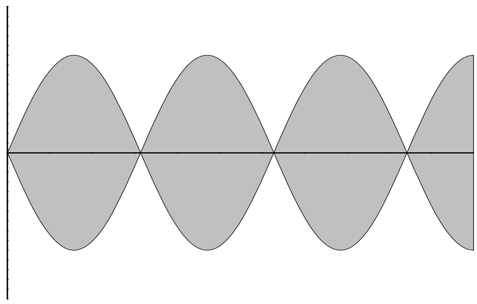
$$y_1 = A\sin\left(kx - \omega t\right),$$
$$y_2 = A\sin\left(kx + \omega t\right).$$

We can write the resultant as:

$$y = y_1 + y_2 = A\left[\sin\left(kx - \omega t\right) + \sin\left(kx - \omega t\right)\right].$$

Using the same trigonometrical equation as before we obtain:

$$y = 2A\sin kx \cdot \cos \omega t\,.$$



This is the equation of a standing wave. Notice that a particle at any particular point *x* executes simple harmonic motion as time goes on, and that all particles vibrate with the same frequency.

In a traveling wave each particle of the string vibrates with the same amplitude. Characteristic of a standing wave, however, is the fact that the amplitude is not the same for different particles but varies with the location *x* of the particle. The amplitude is $2A\sin kx,$ and it has maximum value of *2A* at positions where:

$$kx = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots etc$$

or

$$x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \ldots etc\,.$$

These points are called antinodes and are separated or spaced one-half wavelength apart.

The amplitude has a minimum value of zero at positions where:

$$kx = \pi, 2\pi, 3\pi, \ldots etc$$

or

$$x = \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, \ldots etc\,.$$

These points are called nodes and are spaced one-half wave length apart. The separation between a node and the adjacent antinodes is one-quarter wave length. The superposition of an incident wave and a reflected wave, being the sum of two waves traveling in opposite directions, will give rise to a standing wave.