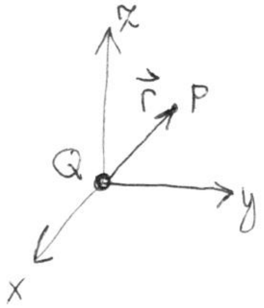
Ponttöltés potenciálja r távolságban:



$$U = \frac{kQ}{r}$$

$$r = \sqrt{x^2+y^2+z^2}$$

Ált skalártér $\varphi(\vec{r})$, vektortér $\vec{v}(\vec{r})$

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

$$\operatorname{grad}\varphi = \begin{pmatrix} \frac{\partial \varphi}{\partial x} \\ \frac{\partial \varphi}{\partial y} \\ \frac{\partial \varphi}{\partial z} \end{pmatrix} \quad (\nabla\varphi)$$

$$\operatorname{div}\vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \quad (\nabla \cdot \vec{v})$$

$$(\nabla \times \vec{v}) = \operatorname{rot}\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

$$\operatorname{rot}\vec{v} = \begin{pmatrix} \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \\ \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \\ \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \end{pmatrix}$$

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial r}\frac{\partial r}{\partial x} = -\frac{kQ}{r^2}\frac{\cancel{2}x}{\underbrace{\cancel{2}\sqrt{x^2+y^2+z^2}}_{r}} = -\frac{kQ}{r^3}x$$

1.) $$\vec{E}(\vec{r}) = -\operatorname{grad}U = -\begin{pmatrix} \frac{\partial kQ/r}{\partial x} \\ \frac{\partial kQ/r}{\partial y} \\ \frac{\partial kQ/r}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{kQ}{r^3}x \\ \frac{kQ}{r^3}y \\ \frac{kQ}{r^3}z \end{pmatrix} = \frac{kQ}{r^3}\vec{r} = \frac{kQ}{r^2}\vec{e}_r \Rightarrow$$

Coulomb-törvény
$\vec{F} = q\vec{E}$
$\vec{F} = k\frac{Qq}{r^2}\vec{e}_r$ ✓

2.a)

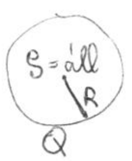
⊛ $$\operatorname{div}\vec{E} = \nabla\cdot\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\partial E_x}{\partial r}\frac{\partial r}{\partial x} + \ldots = -\frac{3kQ}{r^4}\left(\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r}\right) + \frac{3kQ}{r^3} =$$

$$= \frac{3kQ}{r^3}\left(1 - \frac{x^2+y^2+z^2}{r^2}\right) = \frac{3kQ}{r^3}\left(1 - \frac{r^2}{r^2}\right) = 0$$ (ponttöltés $r=0$-ban, $\varrho = 0$ ✓ máshol)

3.) $$\operatorname{rot}\vec{E} = \nabla\times\vec{E} = \begin{pmatrix} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \end{pmatrix} = \begin{pmatrix} -\frac{3kQ}{r^4}\frac{yz}{r} + \frac{3kQ}{r^4}\frac{zy}{r} \\ -\frac{3kQ}{r^4}\frac{zx}{r} + \frac{3kQ}{r^4}\frac{xz}{r} \\ -\frac{3kQ}{r^4}\frac{xy}{r} + \frac{3kQ}{r^4}\frac{yx}{r} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0$$ ✓

2.b)

⊛ másik eset



$r<R$ eset

$$D\cdot 4r^2\pi = \varrho\frac{4}{3}r^3\pi$$

$$D = \frac{\varrho r}{3}$$

$$\vec{D} = \frac{\varrho}{3}\vec{r} = \frac{\varrho}{3}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\operatorname{div}\vec{D} = \nabla\cdot\vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} =$$

$$= \frac{\varrho}{3} + \frac{\varrho}{3} + \frac{\varrho}{3} = \underline{\underline{\varrho}}$$ ✓

$$\mathrm{rot\,grad}\,U = \nabla\times(\nabla U) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} \end{vmatrix} = \underbrace{\left(\frac{\partial^2 U}{\partial z\partial y} - \frac{\partial^2 U}{\partial y\partial z}\right)}_{0}\vec{i} + \underbrace{\left(\frac{\partial^2 U}{\partial x\partial z} - \frac{\partial^2 U}{\partial z\partial x}\right)}_{0}\vec{j} + \underbrace{\left(\frac{\partial^2 U}{\partial y\partial x} - \frac{\partial^2 U}{\partial x\partial y}\right)}_{0}\vec{k}$$

$\mathrm{rotgrad}\,\varphi(\vec{r}) = 0$ minden φ-re!

$$\mathrm{div\,rot}\,\vec{v} = \nabla\cdot(\nabla\times\vec{v}) = \frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) =$$

$$= \frac{\partial^2 v_z}{\partial x\partial y} - \frac{\partial^2 v_y}{\partial x\partial z} + \frac{\partial^2 v_x}{\partial y\partial z} - \frac{\partial^2 v_z}{\partial y\partial x} + \frac{\partial^2 v_y}{\partial z\partial x} - \frac{\partial^2 v_x}{\partial z\partial y} = 0$$

$\mathrm{div\,rot}\,\vec{v} = 0$ minden $\vec{v}$-re!

$$\underbrace{\mathrm{div}\,\mathrm{grad}\,U}_{(-\vec{E})} = \Delta U = \nabla\cdot(\nabla U) = \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial U}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial U}{\partial z}\right) = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}$$

Laplace-operátor $\quad$ ($\Delta U = -\frac{\varrho}{\varepsilon}$ Poisson-egyenlet)

$$\mathrm{rotrot}\,\vec{v} = \nabla\times(\nabla\times\vec{v}) = \nabla(\nabla\cdot\vec{v}) - (\nabla\cdot\nabla)\vec{v} = \mathrm{graddiv}\,\vec{v} - \Delta\vec{v}$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$$

$$\vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = (b_y c_z - b_z c_y)\vec{i} + (b_z c_x - b_x c_z)\vec{j} + (b_x c_y - b_y c_x)\vec{k}$$

$$\vec{a}\times(\vec{b}\times\vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_y c_z - b_z c_y & b_z c_x - b_x c_z & b_x c_y - b_y c_x \end{vmatrix} = \begin{pmatrix} a_y(b_x c_y - b_y c_x) - a_z(b_z c_x - b_x c_z) \\ a_z(b_y c_z - b_z c_y) - a_x(b_x c_y - b_y c_x) \\ a_x(b_z c_x - b_x c_z) - a_y(b_y c_z - b_z c_y) \end{pmatrix} =$$

$$= \begin{pmatrix} b_x(\overset{+a_x b_x c_x}{a_y c_y + a_z c_z}) - c_x(\overset{-a_x b_x c_x}{a_y b_y + a_z b_z}) \\ b_y(a_z c_z + a_x c_x) - c_y(a_z b_z + a_x b_x) \\ b_z(\underset{+a_z b_z c_z}{a_x c_x + a_y c_y}) - c_z(\underset{-a_z b_z c_z}{a_x b_x + a_y b_y}) \end{pmatrix} = \begin{pmatrix} b_x(a_x c_x + a_y c_y + a_z c_z) - c_x(a_x b_x + a_y b_y + a_z b_z) \\ b_y(a_x c_x + a_y b_y + a_z b_z) - c_y(a_x b_x + a_y b_y + a_z b_z) \\ b_z(a_x c_x + a_y c_y + a_z c_z) - c_z(a_x b_x + a_y b_y + a_z b_z) \end{pmatrix} =$$

$$= \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$$